Some Tetration/Exponentiation Formulas:

A note on mathematical notation:

Successive exponentiation versus tetration;

$$(((x)^a)^b)^c = (x)^{abc} \qquad x^{a^{b^c}} = x^{\left(a^{\left(b^c\right)}\right)}$$

Successive exponentiation;

$$(((x)^a)^b)^c = start\ at\ x^a\ and\ go\ to\ the\ right = (x)^{abc}$$

Tetration;

$$x^{a^{b^c}} = start\ at\ b^c\ and\ move\ to\ the\ left = x^{\left(a^{\left(b^c\right)}\right)}$$

Let's look at some formulas for tetration:

$$x^y = z$$

$$y, z\ are\ known; solve\ for\ x$$

$$\boxed{x = z^{\frac{1}{y}}}$$

Continue by iterating "x" as the base with the previous left side of the equation as the new exponent:

$$x^{x^y} = z$$

$$\left(x^{x^y}\right)^y = z^y$$

$$(x^y)^{x^y} = z^y$$

$$z^y = (z^u)^{z^u}$$

$$y = uz^u$$

$$yln(z) = [(u)\ln(z)]e^{[(u)\ln(z)]}$$

Remember the Lambert function; designated as ω:

$$x = ye^y \qquad y = \omega(x)$$

$$x = \omega(x)e^{\omega(x)}$$

$$e^{\omega(x)} = \frac{x}{\omega(x)} \qquad e^{-\omega(x)} = \frac{\omega(x)}{x}$$

$$yln(z) = [(u)\ln(z)]e^{[(u)\ln(z)]} \qquad so, \qquad \omega(yln(z)) = uln(z)$$

$$u = \frac{\omega(yln(z))}{\ln(z)}$$

$$(x^y)^{x^y} = (z^u)^{z^u}$$

$$x^y = z^u = z^{\frac{\omega(yln(z))}{\ln(z)}}$$

$$x = (x^y)^{\frac{1}{y}} = \left(z^{\frac{\omega(yln(z))}{\ln(z)}}\right)^{\frac{1}{y}} = z^{\frac{\omega(yln(z))}{yln(z)}}$$

$$x = z^{\frac{\omega(yln(z))}{yln(z)}} = e^{ln(z)\frac{\omega(yln(z))}{yln(z)}} = e^{\frac{\omega(yln(z))}{y}}$$

$$x^{x^y} = z \qquad x = z^{\frac{\omega(yln(z))}{yln(z)}} = e^{\frac{\omega(yln(z))}{y}}$$

<u>Solve for the case where “y” = 1 (x sought, z known):</u>

$$x^{x^y} = z \qquad let\ y = 1 \qquad x^x = z$$

$$x = z^{\frac{\omega(ln(z))}{ln(z)}} = e^{ln(z)\frac{\omega(yln(z))}{ln(z)}} = e^{\omega(ln(z))}$$

$$x^x = z \qquad x = e^{\omega(ln(z))}$$

<u>Note that the value of “y” can be negative; solve for the case where “y” = –1 (x sought, z known):</u>

$$x^{x^y} = z \qquad let\ y = -1 \qquad x^{\frac{1}{x}} = z \qquad ln(z) = \frac{1}{x}ln(x)$$

$$ln(z) = ln(x)e^{lnx^{-1}} = ln(x)e^{-lnx}$$

$$-ln(z) = -ln(x)e^{-lnx} \qquad \omega(-ln(z)) = -ln(x)$$

$$x^{\frac{1}{x}} = z \qquad x = e^{-\omega(-l\ \ (z))}$$

$$x = z^{\frac{\omega(yln(z))}{yln(z)}} = z^{\frac{\omega(-l\ \ (z))}{-ln(z)}} = e^{ln(z)\frac{\omega(-ln(z))}{-l\ \ (z)}} = e^{-\omega(-ln(z))}$$

Some Additional Tetration Formulas (find x with z & y known):

$$x^{x^{(x+1)}} = u^u = (x^x)^{(x^x)} = z \qquad u = x^x$$

$$u^u = z \qquad u = e^{\omega(ln(z))} \qquad x^x = u$$

$$x = e^{\omega(ln(u))} = e^{\omega\left(ln\left(e^{\omega(ln(z))}\right)\right)} = e^{\omega(\omega(\ln(z)))}$$

$$x^{x^{(x+1)}} = z \qquad x = e^{\omega(\omega(\ln(z)))}$$

$$x^{x^{(x^y+1)}} = u^{u^y} = z \qquad u = x^x \qquad x = e^{\omega(ln(u))}$$

$$u = z^{\frac{\omega(yln(z))}{yln(z)}} = e^{\frac{\omega(yln(z))}{y}} \qquad x = e^{\omega\left(\frac{\omega(yln(z))}{y}\right)}$$

Infinite Tetration:

$$Consider\ the\ infinite\ tetration \quad x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}} = z \qquad x\ is\ known, find\ z$$

$$x^z = z \qquad x = z^{\frac{1}{z}} \qquad ln\,(x) = ln\,(z)e^{-ln\,(z)}$$

$$-l\,n(x) = -l\,n(z)\,e^{-\ln(z)} \qquad \omega(-\ln(x)) = -\ln\,(z)$$

$$ln(z) = -\omega(-\,ln(x)) \qquad z = e^{-\omega(-\,ln(x))}$$

$$Note\ that\ infinite\ tetration\ converges\ for \quad 0 < x \leq e^{\frac{1}{e}}$$

$$For\ 0 < x \leq 1 \quad z\ ranges\ from\ infintesimally\ above\ zero\ to\ one$$

$$For\ 1 < x \leq e^{\frac{1}{e}} \quad z\ has\ two\ values$$

$$one\ value\ 1 < z \leq e$$

$$one\ value\ e \leq z < \infty$$

$$For\ infinite\ exponentiation \ \ (((x)^x)^x)\ldots)^x = z \quad x\ known, find\ z$$

$$For\ x > 1, z = \infty \qquad for\ x = 1, z = 1 \qquad for\ x < 1, z = 1$$

<u>Solve x^y = z for the case where y = z (eigenlogs):</u>

$$x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{y}}}}}}}} = y \qquad any\ number\ of\ "x"\ variables\ up\ to\ infinity$$

$$x^y = y \qquad x = y^{\frac{1}{y}} \qquad lg_x(y) = y \qquad y = Eigenlog \qquad x = Eigenbase$$

$$y = x^y = x^{x^y} = x^{x^{x^y}} = \cdots = x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{y}}}}}}}} \qquad x = y^{\frac{1}{y}} = y^{\frac{\omega(yln(y))}{yln(y)}}$$

$$\frac{\omega(yln(y))}{yln(y)} = \frac{1}{y} \qquad ln(y) = \omega(yln(y))$$

$$yln(y) = \omega(yln(y))e^{\omega(yln(y))} = ln(y)e^{ln(y)} = yln(y)$$

To find the eigenbase “x” (real or complex) for a given eigenlog (real or complex) “y”:

$$x = y^{\frac{1}{y}}$$

To find the eigenvalue “y” (real or complex) for a given eigenbase (real or complex) “x”:

$$\frac{1}{y}\ln(y) = \ln(x)$$

$$ln(y)e^{ln\left(\frac{1}{y}\right)} = ln(y)e^{-l\ \ (y)} = ln(x)$$

$$-ln(y)e^{-\ln(y)} = -ln(x)$$

$$-ln(y) = \omega(-ln(x)) \qquad ln(y) = -\omega(-ln(x))$$

As derived above for the “y = – 1” case in the x^x^y = z tetration formula:

$$\boxed{x = y^{\frac{1}{y}} \qquad y = e^{-\omega(-l\ \ (x))}}$$

Note that the eigenlog can be multivalued for a given eigenbase. The above formula for the eigenlog is difficult to use (*i.e.*, it is difficult to employ this formula with an online Lambert function calculator) for values of eigenbase “x” LTET to zero or greater than exp(1/e) (≈ 1.444667861), as eigenlogs are complex outside this range. It is also difficult to use the formula for complex values of the eigenbase “x”. An easier way to get the eigenlog “y” for an arbitrary value of the eigenbase “x” is via taking an “infinite number of logs”. Of course, one does not actually take the log an infinite number of times, but rather the log is taken until the value of the eigenlog converges to an acceptable degree of accuracy.

$$y = \lim_{take\ lg\ \infty\ times} lg_x\left(lg_x \ldots \left(lg_x\left(lg_x\left(lg_x(lg_x(any\ number))\right)\right)\right)\right)$$

$$lg_x(y) = \frac{ln(y)}{ln(x)} \qquad lg_{a+bi}(c+di) = \frac{\ln\left(\sqrt{c^2+d^2}\right) + \arctan\left(\frac{d}{c}\right)i}{\ln\left(\sqrt{a^2+b^2}\right) + \arctan\left(\frac{b}{a}\right)i}$$

Here are a few examples carried out with an EXCEL spreadsheet;

For the eigenbase “x = 1.3^(1/1.3)” (≈1.223626) starting with seed number “2”, the eigenlog “y = 12.5245726091972”.

iteration	seed number then iteration	complex log to the base “x”	“x”
0	2	3.4345048345545	1.22362610172251
1	3.4345048345545	6.11376932092997	1.22362610172251
2	6.11376932092997	8.97113997524887	1.22362610172251
3	8.97113997524887	10.8712083483919	1.22362610172251
4	10.8712083483919	11.823078201692	1.22362610172251
5	11.823078201692	12.2389740390701	1.22362610172251
6	12.2389740390701	12.4102765893532	1.22362610172251
7	12.4102765893532	12.4791474906414	1.22362610172251
8	12.4791474906414	12.5065689582083	1.22362610172251
9	12.5065689582083	12.5174449213725	1.22362610172251
10	12.5174449213725	12.5217519667738	1.22362610172251
11	12.5217519667738	12.5234565873466	1.22362610172251
12	12.5234565873466	12.5241310714842	1.22362610172251
13	12.5241310714842	12.5243979259666	1.22362610172251
14	12.5243979259666	12.5245035009306	1.22362610172251
15	12.5245035009306	12.5245452686656	1.22362610172251
16	12.5245452686656	12.5245617927861	1.22362610172251
17	12.5245617927861	12.5245683300313	1.22362610172251
18	12.5245683300313	12.524570916283	1.22362610172251
19	12.524570916283	12.5245719394503	1.22362610172251
20	12.5245719394503	12.5245723442334	1.22362610172251
21	12.5245723442334	12.5245725043728	1.22362610172251
22	12.5245725043728	12.5245725677268	1.22362610172251
23	12.5245725677268	12.5245725927908	1.22362610172251
24	12.5245725927908	12.5245726027066	1.22362610172251
25	12.5245726027066	12.5245726066294	1.22362610172251
26	12.5245726066294	12.5245726081814	1.22362610172251
27	12.5245726081814	12.5245726087953	1.22362610172251
28	12.5245726087953	12.5245726090382	1.22362610172251
29	12.5245726090382	12.5245726091343	1.22362610172251
30	12.5245726091343	12.5245726091724	1.22362610172251
31	12.5245726091724	12.5245726091874	1.22362610172251
32	12.5245726091874	12.5245726091934	1.22362610172251
33	12.5245726091934	12.5245726091957	1.22362610172251
34	12.5245726091957	12.5245726091966	1.22362610172251

35	12.5245726091966	12.524572609197	1.22362610172251
36	12.524572609197	12.5245726091971	1.22362610172251
37	12.5245726091971	12.5245726091972	1.22362610172251
38	12.5245726091972	12.5245726091972	1.22362610172251
39	12.5245726091972	12.5245726091972	1.22362610172251
40	12.5245726091972	12.5245726091972	1.22362610172251
41	12.5245726091972	12.5245726091972	1.22362610172251
42	12.5245726091972	12.5245726091972	1.22362610172251
43	12.5245726091972	12.5245726091972	1.22362610172251
44	12.5245726091972	12.5245726091972	1.22362610172251
45	12.5245726091972	12.5245726091972	1.22362610172251
46	12.5245726091972	12.5245726091972	1.22362610172251
47	12.5245726091972	12.5245726091972	1.22362610172251
48	12.5245726091972	12.5245726091972	1.22362610172251
49	12.5245726091972	12.5245726091972	1.22362610172251
50	12.5245726091972	12.5245726091972	1.22362610172251

The eigenlog has two values for the eigenbase "x = 1.3^(1/1.3) = 12.5245726091972^(1/12.5245726091972)" (≈1.223626), with the two values of the eigenlog being "y = 12.5245726091972 & y = 1.3". Note that the infinite log method converges to the greater value of the eigenlog (y = 12.5245726091972). The lesser value of the eigenlog (y = 1.3) is, using this method, "unstable" (see Table below).

iteration	seed number then iteration	complex log to the base "x"	"x"
0	1.3	1.30000000000001	1.22362610172251
1	1.30000000000001	1.30000000000005	1.22362610172251
2	1.30000000000005	1.3000000000002	1.22362610172251
3	1.3000000000002	1.30000000000077	1.22362610172251
4	1.30000000000077	1.30000000000294	1.22362610172251
5	1.30000000000294	1.30000000001121	1.22362610172251
6	1.30000000001121	1.30000000004273	1.22362610172251
7	1.30000000004273	1.30000000016287	1.22362610172251
8	1.30000000016287	1.30000000062079	1.22362610172251
9	1.30000000062079	1.30000000236615	1.22362610172251
10	1.30000000236615	1.30000000901857	1.22362610172251
11	1.30000000901857	1.30000003437424	1.22362610172251
12	1.30000003437424	1.30000013101724	1.22362610172251
13	1.30000013101724	1.30000049937149	1.22362610172251
14	1.30000049937149	1.30000190335142	1.22362610172251
15	1.30000190335142	1.30000725460852	1.22362610172251
16	1.30000725460852	1.30002765082468	1.22362610172251
17	1.30002765082468	1.30010538985055	1.22362610172251
18	1.30010538985055	1.30040167657385	1.22362610172251
19	1.30040167657385	1.30153075165185	1.22362610172251
20	1.30153075165185	1.30583101944463	1.22362610172251
21	1.30583101944463	1.32217520440022	1.22362610172251
22	1.32217520440022	1.38380789748211	1.22362610172251
23	1.38380789748211	1.60955898308249	1.22362610172251
24	1.60955898308249	2.35835579419868	1.22362610172251
25	2.35835579419868	4.25116615930441	1.22362610172251
26	4.25116615930441	7.17076062267108	1.22362610172251
27	7.17076062267108	9.76129602920359	1.22362610172251
28	9.76129602920359	11.2894671152886	1.22362610172251

29	11.2894671152886	12.0101387928577	1.22362610172251
30	12.0101387928577	12.3167557013334	1.22362610172251
31	12.3167557013334	12.4416668191061	1.22362610172251
32	12.4416668191061	12.4916645708754	1.22362610172251
33	12.4916645708754	12.511536471903	1.22362610172251
34	12.511536471903	12.519412596188	1.22362610172251
35	12.519412596188	12.5225307958859	1.22362610172251
36	12.5225307958859	12.5237647657814	1.22362610172251
37	12.5237647657814	12.5242530017044	1.22362610172251
38	12.5242530017044	12.5244461651904	1.22362610172251
39	12.5244461651904	12.5245225854527	1.22362610172251
40	12.5245225854527	12.5245528188771	1.22362610172251
41	12.5245528188771	12.5245647797893	1.22362610172251
42	12.5245647797893	12.5245695117436	1.22362610172251
43	12.5245695117436	12.5245713837895	1.22362610172251
44	12.5245713837895	12.5245721244042	1.22362610172251
45	12.5245721244042	12.5245724174045	1.22362610172251
46	12.5245724174045	12.5245725333206	1.22362610172251
47	12.5245725333206	12.5245725791791	1.22362610172251
48	12.5245725791791	12.5245725973215	1.22362610172251
49	12.5245725973215	12.524572604499	1.22362610172251
50	12.524572604499	12.5245726073385	1.22362610172251
51	12.5245726073385	12.5245726084619	1.22362610172251
52	12.5245726084619	12.5245726089063	1.22362610172251
53	12.5245726089063	12.5245726090821	1.22362610172251
54	12.5245726090821	12.5245726091517	1.22362610172251
55	12.5245726091517	12.5245726091792	1.22362610172251
56	12.5245726091792	12.5245726091901	1.22362610172251
57	12.5245726091901	12.5245726091944	1.22362610172251
58	12.5245726091944	12.5245726091961	1.22362610172251
59	12.5245726091961	12.5245726091968	1.22362610172251
60	12.5245726091968	12.5245726091971	1.22362610172251
61	12.5245726091971	12.5245726091972	1.22362610172251
62	12.5245726091972	12.5245726091972	1.22362610172251
63	12.5245726091972	12.5245726091972	1.22362610172251
64	12.5245726091972	12.5245726091972	1.22362610172251
65	12.5245726091972	12.5245726091972	1.22362610172251
66	12.5245726091972	12.5245726091972	1.22362610172251
67	12.5245726091972	12.5245726091972	1.22362610172251
68	12.5245726091972	12.5245726091972	1.22362610172251
69	12.5245726091972	12.5245726091972	1.22362610172251
70	12.5245726091972	12.5245726091972	1.22362610172251

For the eigenbase “x = 5” with the seed number “3”, the eigenlog is “–0.0107508384725545 + 0.982787226228601i ”.

iteration	seed number then iteration	complex log to the base “x”	“x”
0	3	0.682606194485986	5
1	0.682606194485986	-0.237248771611776	5
2	-0.237248771611776	-0.893881030142234+1.95198126583117i	5
3	-0.893881030142234+1.95198126583117i	0.474720411488325+1.24281251592805i	5
4	0.474720411488325+1.24281251592805i	0.177374662890601+0.749284381156858i	5
5	0.177374662890601+0.749284381156858i	-0.162400898488729+0.831563529867015i	5

6	-0.162400898488729+0.831563529867015i	-0.102975100030223+1.09582642364007i	5
7	-0.102975100030223+1.09582642364007i	0.0595888962365977+1.03420668614419i	5
8	0.0595888962365977+1.03420668614419i	0.0219280359190245+0.94023010943311i	5
9	0.0219280359190245+0.94023010943311i	-0.0381243366333657+0.961502493030239i	5
10	-0.0381243366333657+0.961502493030239i	-0.0239043987925318+1.00061415714902i	5
11	-0.0239043987925318+1.00061415714902i	0.000558733423430659+0.990831331859716i	5
12	0.000558733423430659+0.990831331859716i	-0.0057229918673688 9+0.975640259903424i	5
13	-0.00572299186736889+0.975640259903424i	-0.0153122667799287+0.979635269304567i	5
14	-0.0153122667799287+0.979635269304567i	-0.0127080442355869+0.985701667094535i	5
15	-0.0127080442355869+0.985701667094535i	-0.00889654645715653+0.984000677606869i	5
16	-0.00889654645715653+0.984000677606869i	-0.00999592665672596+0.981608092969755i	5
17	-0.00999592665672596+0.981608092969755i	-0.0115017140796799+0.982317601702889i	5
18	-0.0115017140796799+0.982317601702889i	-0.0110423997773777+0.983265357709695i	5
19	-0.0110423997773777+0.983265357709695i	-0.0104466236865475+0.982968139315513i	5
20	-0.0104466236865475+0.982968139315513i	-0.0106385594739116+0.982593703379883i	5
21	-0.0106385594739116+0.982593703379883i	-0.0108739570086612+0.982717574439307i	5
22	-0.0108739570086612+0.982717574439307i	-0.0107940134495326+0.982865541654301i	5
23	-0.0107940134495326+0.982865541654301i	-0.010701034903731+0.98281397522181i	5
24	-0.010701034903731+0.98281397522181i	-0.0107342731332923+0.982755559198754i	5
25	-0.0107342731332923+0.982755559198754i	-0.010770971308521+0.982776973237975i	5
26	-0.010770971308521+0.982776973237975i	-0.0107571804938508+0.982800024060597i	5
27	-0.0107571804938508+0.982800024060597i	-0.01074270461713+0.982791146734115i	5
28	-0.01074270461713+0.982791146734115i	-0.0107484163877898+0.982782057390571i	5
29	-0.0107484163877898+0.982782057390571i	-0.010754122665369+0.982785730863318i	5
30	-0.010754122665369+0.982785730863318i	-0.0107517610417378+0.982789312646837i	5
31	-0.0107517610417378+0.982789312646837i	-0.0107495131825151+0.982787794993903i	5
32	-0.0107495131825151+0.982787794993903i	-0.0107504880964171+0.982786384525185i	5
33	-0.0107504880964171+0.982786384525185i	-0.010751372971286+0.982787010561562i	5
34	-0.010751372971286+0.982787010561562i	-0.0107509711084608+0.982787565598854i	5
35	-0.0107509711084608+0.982787565598854i	-0.010750623025378+0.982787307726458i	5
36	-0.010750623025378+0.982787307726458i	-0.0107507884441983+0.982787089471944i	5
37	-0.0107507884441983+0.982787089471944i	-0.0107509252680683+0.982787195549276i	5
38	-0.0107509252680683+0.982787195549276i	-0.0107508572660292+0.982787281307794i	5
39	-0.0107508572660292+0.982787281307794i	-0.0107508035247534+0.982787237727858i	5
40	-0.0107508035247534+0.982787237727858i	-0.0107508314450635+0.982787204057128i	5
41	-0.0107508314450635+0.982787204057128i	-0.0107508525366586+0.982787221939545i	5
42	-0.0107508525366586+0.982787221939545i	-0.0107508410865905+0.982787235148759i	5
43	-0.0107508410865905+0.982787235148759i	-0.0107508328156763+0.982787227819356i	5
44	-0.0107508328156763+0.982787227819356i	-0.0107508375060901+0.982787222641654i	5
45	-0.0107508375060901+0.982787222641654i	-0.0107508407466959+0.982787225642459i	5
46	-0.0107508407466959+0.982787225642459i	-0.0107508388273534+0.982787227670229i	5
47	-0.0107508388273534+0.982787227670229i	-0.0107508375587883+0.982787226442912i	5
48	-0.0107508375587883+0.982787226442912i	-0.0107508383433985+0.982787225649486i	5
49	-0.0107508383433985+0.982787225649486i	-0.0107508388395304+0.982787226150959i	5
50	-0.0107508388395304+0.982787226150959i	-0.0107508385190976+0.982787226461117i	5
51	-0.0107508385190976+0.982787226461117i	-0.0107508383252497+0.982787226256412i	5
52	-0.0107508383252497+0.982787226256412i	-0.0107508384559928+0.982787226135289i	5
53	-0.0107508384559928+0.982787226135289i	-0.0107508385316556+0.982787226218772i	5
54	-0.0107508385316556+0.982787226218772i	-0.0107508384783592+0.982787226266025i	5
55	-0.0107508384783592+0.982787226266025i	-0.0107508384488573+0.982787226232006i	5
56	-0.0107508384488573+0.982787226232006i	-0.010750838470566+0.982787226213596i	5

57	-0.010750838470566+0.982787226213596i	-0.0107508384820536+0.982787226227446i	5
58	-0.0107508384820536+0.982787226227446i	-0.0107508384732191+0.98278722623461i	5
59	-0.0107508384732191+0.98278722623461i	-0.0107508384687515+0.982787226228974i	5
60	-0.0107508384687515+0.982787226228974i	-0.0107508384723452+0.982787226226191i	5
61	-0.0107508384723452+0.982787226226191i	-0.0107508384740796+0.982787226228483i	5
62	-0.0107508384740796+0.982787226228483i	-0.0107508384726187+0.982787226229565i	5
63	-0.0107508384726187+0.982787226229565i	-0.0107508384719448+0.982787226228632i	5
64	-0.0107508384719448+0.982787226228632i	-0.0107508384725392+0.98278722622821i	5
65	-0.0107508384725392+0.98278722622821i	-0.0107508384728019+0.982787226228589i	5
66	-0.0107508384728019+0.982787226228589i	-0.0107508384725604+0.982787226228757i	5
67	-0.0107508384725604+0.982787226228757i	-0.0107508384724559+0.982787226228601i	5
68	-0.0107508384724559+0.982787226228601i	-0.0107508384725553+0.982787226228533i	5
69	-0.0107508384725553+0.982787226228533i	-0.0107508384725976+0.982787226228601i	5
70	-0.0107508384725976+0.982787226228601i	-0.0107508384725543+0.982787226228626i	5
71	-0.0107508384725543+0.982787226228626i	-0.0107508384725388+0.982787226228595i	5
72	-0.0107508384725388+0.982787226228595i	-0.0107508384725585+0.982787226228589i	5
73	-0.0107508384725585+0.982787226228589i	-0.0107508384725621+0.982787226228601i	5
74	-0.0107508384725621+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
75	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
76	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
77	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
78	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
79	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
80	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5

For the eigenbase “x = 5” starting with the seed number “1 + i”, the iterated value of the eigenlog is “–0.0107508384725545 + 0.982787226228601i ”. The use of many different seed numbers for the eigenbase “x = 5” all result in iteration to the same eigenlog (i.e., “y = –0.0107508384725545 + 0.982787226228601i”).

iteration	seed number then iteration	complex log to the base “x”	“x”
0	1+i	0.215338279036697+0.487995316457793i	5
1	0.215338279036697+0.487995316457793i	-0.390504925765449+0.717781611123158i	5
2	-0.390504925765449+0.717781611123158i	-0.125481544856658+1.28557643484002i	5
3	-0.125481544856658+1.28557643484002i	0.159029583864702+1.03644590612062i	5
4	0.159029583864702+1.03644590612062i	0.0294715013975128+0.881392377505953i	5
5	0.0294715013975128+0.881392377505953i	-0.0780978595198164+0.955222523383131i	5
6	-0.0780978595198164+0.955222523383131i	-0.0263942000575607+1.02667750131383i	5
7	-0.0263942000575607+1.02667750131383i	0.0165636786272795+0.991960620595919i	5
8	0.0165636786272795+0.991960620595919i	-0.0049287260956206+0.965616596265421i	5
9	-0.0049287260956206+0.965616596265421i	-0.0217314350336133+0.97916203979011i	5
10	-0.0217314350336133+0.97916203979011i	-0.0129311669406536+0.989778221585156i	5
11	-0.0129311669406536+0.989778221585156i	-0.00633080866964012+0.984107732791272i	5
12	-0.00633080866964012+0.984107732791272i	-0.00994086920270477+0.979987652955784i	5
13	-0.00994086920270477+0.979987652955784i	-0.0125285106335631+0.98229315872406i	5
14	-0.0125285106335631+0.98229315872406i	-0.0110499153966111+0.98391492639313i	5
15	-0.0110499153966111+0.98391492639313i	-0.0100362888809663+0.982968278696776i	5
16	-0.0100362888809663+0.982968278696776i	-0.0106411734700384+0.982334357683441i	5
17	-0.0106411734700384+0.982334357683441i	-0.0110379366078595+0.982721003458685i	5
18	-0.0110379366078595+0.982721003458685i	-0.010790689964431+0.982969182482886i	5
19	-0.010790689964431+0.982969182482886i	-0.0106355510322416+0.982811155328216i	5
20	-0.0106355510322416+0.982811155328216i	-0.0107365049893387+0.982714184466343i	5
21	-0.0107365049893387+0.982714184466343i	-0.0107971119830416+0.982778669891353i	5

22	-0.0107971119830416+0.982778669891353i	-0.0107559266329321+0.982816537042728i	5
23	-0.0107559266329321+0.982816537042728i	-0.01073227497551+0.982790239889304i	5
24	-0.01073227497551+0.982790239889304i	-0.0107490616719356+0.982775470665641i	5
25	-0.0107490616719356+0.982775470665641i	-0.0107582819821898+0.98278618431735i	5
26	-0.0107582819821898+0.98278618431735i	-0.0107514456178653+0.982791938789447i	5
27	-0.0107514456178653+0.982791938789447i	-0.0107478552758758+0.982787577440932i	5
28	-0.0107478552758758+0.982787577440932i	-0.0107506370828684+0.982785337998123i	5
29	-0.0107506370828684+0.982785337998123i	-0.0107520334951662+0.982787111978863i	5
30	-0.0107520334951662+0.982787111978863i	-0.0107509024304229+0.9827879824421li	5
31	-0.0107509024304229+0.98278798244211i	-0.010750359996522+0.982787261429717i	5
32	-0.010750359996522+0.982787261429717i	-0.0107508195290944+0.982786923520634i	5
33	-0.0107508195290944+0.982786923520634i	-0.010751029957848+0.982787216346897i	5
34	-0.010751029957848+0.982787216346897i	-0.0107508433950421+0.98278734734273i	5
35	-0.0107508433950421+0.98278734734273i	-0.0107507618772527+0.982787228502793i	5
36	-0.0107507618772527+0.982787228502793i	-0.0107508375646042+0.982787177793804i	5
37	-0.0107508375646042+0.982787177793804i	-0.0107508690964792+0.982787225989574i	5
38	-0.0107508690964792+0.982787225989574i	-0.0107508384118863+0.982787245588907i	5
39	-0.0107508384118863+0.982787245588907i	-0.0107508262345215+0.982787226056367i	5
40	-0.0107508262345215+0.982787226056367i	-0.0107508386660578+0.982787218493621i	5
41	-0.0107508386660578+0.982787218493621i	-0.0107508433608184+0.982787226404408i	5
42	-0.0107508433608184+0.982787226404408i	-0.010750838327617+0.982787229317456i	5
43	-0.010750838327617+0.982787229317456i	-0.0107508365209635+0.982787226115618i	5
44	-0.0107508365209635+0.982787226115618i	-0.0107508385574713+0.982787224995699i	5
45	-0.0107508385574713+0.982787224995699i	-0.0107508392513359+0.982787226290803i	5
46	-0.0107508392513359+0.982787226290803i	-0.0107508384278489+0.982787226720469i	5
47	-0.0107508384278489+0.982787226720469i	-0.0107508381619336+0.982787226196938i	5
48	-0.0107508381619336+0.982787226196938i	-0.0107508384947181+0.982787226032458i	5
49	-0.0107508384947181+0.982787226032458i	-0.0107508385963914+0.982787226243967i	5
50	-0.0107508385963914+0.982787226243967i	-0.0107508384619848+0.982787226306772i	5
51	-0.0107508384619848+0.982787226306772i	-0.0107508384232126+0.982787226221375i	5
52	-0.0107508384232126+0.982787226221375i	-0.0107508384774637+0.98278722619746i	5
53	-0.0107508384774637+0.98278722619746i	-0.0107508384922063+0.982787226231919i	5
54	-0.0107508384922063+0.982787226231919i	-0.0107508384703212+0.982787226240997i	5
55	-0.0107508384703212+0.982787226240997i	-0.010750838464734+0.982787226227104i	5
56	-0.010750838464734+0.982787226227104i	-0.010750838473555+0.982787226223668i	5
57	-0.010750838473555+0.982787226223668i	-0.0107508384756661+0.982787226229266i	5
58	-0.0107508384756661+0.982787226229266i	-0.0107508384721127+0.982787226230559i	5
59	-0.0107508384721127+0.982787226230559i	-0.0107508384713199+0.982787226228303i	5
60	-0.0107508384713199+0.982787226228303i	-0.0107508384727516+0.982787226227819i	5
61	-0.0107508384727516+0.982787226227819i	-0.0107508384730476+0.982787226228726i	5
62	-0.0107508384730476+0.982787226228726i	-0.0107508384724722+0.982787226228912i	5
63	-0.0107508384724722+0.982787226228912i	-0.0107508384723586+0.982787226228546i	5
64	-0.0107508384723586+0.982787226228546i	-0.0107508384725908+0.982787226228477i	5
65	-0.0107508384725908+0.982787226228477i	-0.0107508384726328+0.98278722622862i	5
66	-0.0107508384726328+0.98278722622862i	-0.0107508384725421+0.982787226228645i	5
67	-0.0107508384725421+0.982787226228645i	-0.0107508384725269+0.982787226228589i	5
68	-0.0107508384725269+0.982787226228589i	-0.0107508384725624+0.982787226228583i	5
69	-0.0107508384725624+0.982787226228583i	-0.0107508384725659+0.982787226228601i	5
70	-0.0107508384725659+0.982787226228601i	-0.0107508384725545+0.982787226228608i	5
71	-0.0107508384725545+0.982787226228608i	-0.0107508384725501+0.982787226228595i	5
72	-0.0107508384725501+0.982787226228595i	-0.0107508384725584+0.982787226228595i	5

73	-0.0107508384725584+0.982787226228595i	-0.0107508384725583+0.982787226228601i	5
74	-0.0107508384725583+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
75	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
76	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
77	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
78	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
79	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5
80	-0.0107508384725545+0.982787226228601i	-0.0107508384725545+0.982787226228601i	5

For the eigenbase "x = 3 + 2i" with the seed number "2", the eigenlog is "–0.36452405787166 – 1.27484729025114i"

iteration	seed number then iteration	complex log to the base "x"	"x"
0	2	0.446595782252787-0.204759974643624i	3+2i
1	0.446595782252787-0.204759974643624i	-0.58489981331754-0.0670341775315805i	3+2i
2	-0.58489981331754-0.0670341775315805i	-1.23568160273274-1.79410837523042i	3+2i
3	-1.23568160273274-1.79410837523042i	-0.140523799209821-1.63067518920793i	3+2i
4	-0.140523799209821-1.63067518920793i	-0.171973748618434-1.21299717847489i	3+2i
5	-0.171973748618434-1.21299717847489i	-0.374805080551977-1.16278882563838i	3+2i
6	-0.374805080551977-1.16278882563838i	-0.427117518390044-1.27212477615303i	3+2i
7	-0.427117518390044-1.27212477615303i	-0.370225772279565-1.30764978029219i	3+2i
8	-0.370225772279565-1.30764978029219i	-0.347863915497787-1.28045707451794i	3+2i
9	-0.347863915497787-1.28045707451794i	-0.360162809545055-1.2665270903913i	3+2i
10	-0.360162809545055-1.2665270903913i	-0.368581181146283-1.27185855093874i	3+2i
11	-0.368581181146283-1.27185855093874i	-0.366428382970541-1.27675497409435i	3+2i
12	-0.366428382970541-1.27675497409435i	-0.363669659741691-1.27600596574985i	3+2i
13	-0.363669659741691-1.27600596574985i	-0.363842952762337-1.27448960726217i	3+2i
14	-0.363842952762337-1.27448960726217i	-0.364658110524776-1.27445842123545i	3+2i
15	-0.364658110524776-1.27445842123545i	-0.364740385252326-1.27488680457548i	3+2i
16	-0.364740385252326-1.27488680457548i	-0.364520607795021-1.2749647886425i	3+2i
17	-0.364520607795021-1.2749647886425i	-0.364461690870018-1.27485493999037i	3+2i
18	-0.364461690870018-1.27485493999037i	-0.364514986743545-1.27481494942746i	3+2i
19	-0.364514986743545-1.27481494942746i	-0.364540417181449-1.27483988932287i	3+2i
20	-0.364540417181449-1.27483988932287i	-0.364529287660574-1.27485533866964i	3+2i
21	-0.364529287660574-1.27485533866964i	-0.364520226335001-1.27485070274036i	3+2i
22	-0.364520226335001-1.27485070274036i	-0.364521945664499-1.27484554060024i	3+2i
23	-0.364521945664499-1.27484554060024i	-0.364524812187826-1.27484603300918i	3+2i
24	-0.364524812187826-1.27484603300918i	-0.364524783067762-1.27484758751175i	3+2i
25	-0.364524783067762-1.27484758751175i	-0.364523959248647-1.27484769744268i	3+2i
26	-0.364523959248647-1.27484769744268i	-0.364523834742061-1.27484727096088i	3+2i
27	-0.364523834742061-1.27484727096088i	-0.364524050077408-1.27484717078398i	3+2i
28	-0.364524050077408-1.27484717078398i	-0.364524120375303-1.27484727650118i	3+2i
29	-0.364524120375303-1.27484727650118i	-0.364524070176653-1.27484732217249i	3+2i
30	-0.364524070176653-1.27484732217249i	-0.364524041994933-1.27484729932709i	3+2i
31	-0.364524041994933-1.27484729932709i	-0.364524051795568-1.27484728259282i	3+2i
32	-0.364524051795568-1.27484728259282i	-0.364524061428833-1.27484728642286i	3+2i
33	-0.364524061428833-1.27484728642286i	-0.364524060181473-1.2748472918223i	3+2i
34	-0.364524060181473-1.2748472918223i	-0.364524057227599-1.27484729159842i	3+2i
35	-0.364524057227599-1.27484729159842i	-0.364524057107774-1.27484729001941i	3+2i
36	-0.364524057107774-1.27484729001941i	-0.364524057932536-1.27484728982879i	3+2i
37	-0.364524057932536-1.27484728982879i	-0.364524058099757-1.27484729024926i	3+2i
38	-0.364524058099757-1.27484729024926i	-0.364524057891043-1.27484729037153i	3+2i
39	-0.364524057891043-1.27484729037153i	-0.364524057809602-1.27484729027109i	3+2i

40	-0.364524057809602-1.27484729027109i	-0.364524057856117-1.27484729021996i	3+2i
41	-0.364524057856117-1.27484729021996i	-0.364524057886886-1.27484729024042i	3+2i
42	-0.364524057886886-1.27484729024042i	-0.364524057878552-1.27484729025832i	3+2i
43	-0.364524057878552-1.27484729025832i	-0.364524057868422-1.27484729025537i	3+2i
44	-0.364524057868422-1.27484729025537i	-0.364524057869165-1.27484729024978i	3+2i
45	-0.364524057869165-1.27484729024978i	-0.364524057872177-1.27484729024971i	3+2i
46	-0.364524057872177-1.27484729024971i	-0.364524057872457-1.2748472902513i	3+2i
47	-0.364524057872457-1.2748472902513i	-0.364524057871642-1.27484729025158i	3+2i
48	-0.364524057871642-1.27484729025158i	-0.364524057871428-1.27484729025117i	3+2i
49	-0.364524057871428-1.27484729025117i	-0.364524057871627-1.27484729025103i	3+2i
50	-0.364524057871627-1.27484729025103i	-0.364524057871716-1.27484729025112i	3+2i
51	-0.364524057871716-1.27484729025112i	-0.364524057871677-1.27484729025118i	3+2i
52	-0.364524057871677-1.27484729025118i	-0.364524057871643-1.27484729025116i	3+2i
53	-0.364524057871643-1.27484729025116i	-0.364524057871647-1.27484729025113i	3+2i
54	-0.364524057871647-1.27484729025113i	-0.364524057871664-1.27484729025114i	3+2i
55	-0.364524057871664-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
56	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
57	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
58	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
59	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
60	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
61	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
62	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
63	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
64	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
65	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
66	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
67	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
68	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
69	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
70	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
71	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
72	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
73	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
74	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
75	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
76	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
77	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
78	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
79	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i
80	-0.36452405787166-1.27484729025114i	-0.36452405787166-1.27484729025114i	3+2i

For the eigenbase “x = 3 + 2i” starting with seed number “1 + i”, the eigenlog iterates to the different value of “0.286648594624413 + 0.835758121589514i”. The eigenlog is multivalued for this eigenbase. Note that iteration to a constant value (to 15 decimal places) takes significantly longer for this combination of eigenbase and seed number.

iteration	seed number then iteration	complex log to the base “x”	“x”
0	1+i	0.455309378055471+0.403653243425856i	3+2i
1	0.455309378055471+0.403653243425856i	-0.105820220455332+0.614090295619549i	3+2i
2	-0.105820220455332+0.614090295619549i	0.209689158173768+1.26173482648581i	3+2i
3	0.209689158173768+1.26173482648581i	0.573942827842795+0.833256588499195i	3+2i
4	0.573942827842795+0.833256588499195i	0.29339835189626 8+0.619982042865347i	3+2i

5	0.293398351896268+0.619982042865347i	0.0905351045209481+0.838654113096242i	3+2i
6	0.0905351045209481+0.838654113096242i	0.322619321515897+0.99304800303517i	3+2i
7	0.322619321515897+0.99304800303517i	0.399059014512738+0.796917997821919i	3+2i
8	0.399059014512738+0.796917997821919i	0.25270201430135 8+0.746959871153589i	3+2i
9	0.25270201430135 8+0.746959871153589i	0.214593561053352+0.872059545116103i	3+2i
10	0.214593561053352+0.872059545116103i	0.323482856116888+0.888364351752118i	3+2i
11	0.323482856116888+0.888364351752118i	0.324707038731386+0.803647909741047i	3+2i
12	0.324707038731386+0.803647909741047i	0.258465598179881+0.806902784231238i	3+2i
13	0.258465598179881+0.806902784231238i	0.265679207191763+0.861291668431853i	3+2i
14	0.265679207191763+0.861291668431853i	0.308708444274159+0.849973420639593i	3+2i
15	0.308708444274159+0.849973420639593i	0.296294274123678+0.817319022740774i	3+2i
16	0.296294274123678+0.817319022740774i	0.271090305089687+0.829341982441732i	3+2i
17	0.271090305089687+0.829341982441732i	0.282836227401399+0.848798032413767i	3+2i
18	0.282836227401399+0.848798032413767i	0.297303154557727+0.837703100885734i	3+2i
19	0.297303154557727+0.837703100885734i	0.287396051412218+0.827128102617738i	3+2i
20	0.287396051412218+0.827128102617738i	0.27966868351807+0.835835528847713i	3+2i
21	0.27966868351807+0.835835528847713i	0.287288215881031+0.84132812325171i	3+2i
22	0.287288215881031+0.84132812325171i	0.291029652315745+0.834803730202101i	3+2i
23	0.291029652315745+0.834803730202101i	0.285546639985691+0.832335236829158i	3+2i
24	0.285546639985691+0.832335236829158i	0.283998650373538+0.836912723606059i	3+2i
25	0.283998650373538+0.836912723606059i	0.287784533584854+0.837779315131749i	3+2i
26	0.287784533584854+0.837779315131749i	0.288169787153264+0.834692056977638i	3+2i
27	0.288169787153264+0.834692056977638i	0.285677269286252+0.834627465411487i	3+2i
28	0.285677269286252+0.834627465411487i	0.285823068163048+0.836624501673002i	3+2i
29	0.285823068163048+0.836624501673002i	0.28740607149066+0.836347425730558i	3+2i
30	0.28740607149066+0.836347425730558i	0.287058658547523+0.83510693293667i	3+2i
31	0.287058658547523+0.83510693293667i	0.2860960559969+0.835482411250568i	3+2i
32	0.2860960559969+0.835482411250568i	0.28647260389977+0.836221309432894i	3+2i
33	0.28647260389977+0.836221309432894i	0.28703229058524+0.835861609326905i	3+2i
34	0.28703229058524+0.835861609326905i	0.286700683891718+0.83544364157432i	3+2i
35	0.286700683891718+0.83544364157432i	0.286393372810903+0.835741573652454i	3+2i
36	0.286393372810903+0.835741573652454i	0.286655748213759+0.835963180338214i	3+2i
37	0.286655748213759+0.835963180338214i	0.28681169312563+0.835736069881992i	3+2i
38	0.28681169312563+0.835736069881992i	0.286618009143+0.835629685390484i	3+2i
39	0.286618009143+0.835629685390484i	0.286548500180554+0.835792771502223i	3+2i
40	0.286548500180554+0.835792771502223i	0.28668423355006+0.835835258657655i	3+2i
41	0.28668423355006+0.835835258657655i	0.286707326767594+0.835723528907274i	3+2i
42	0.286707326767594+0.835723528907274i	0.286616303012207+0.835713993543176i	3+2i
43	0.286616303012207+0.835713993543176i	0.286615941765595+0.835787415365018i	3+2i
44	0.286615941765595+0.835787415365018i	0.286674580213623+0.835781851762121i	3+2i
45	0.286674580213623+0.835781851762121i	0.286665466627225+0.835735488520673i	3+2i
46	0.286665466627225+0.835735488520673i	0.286629183547242+0.835746456537506i	3+2i
47	0.286629183547242+0.835746456537506i	0.286640829704471+0.835774546058076i	3+2i
48	0.286640829704471+0.835774546058076i	0.286662324005265+0.83576301113117i	3+2i
49	0.286662324005265+0.83576301113117i	0.286651403834547+0.835746772661559i	3+2i
50	0.286651403834547+0.835746772661559i	0.286639311513389+0.835756783348208i	3+2i
51	0.286639311513389+0.835756783348208i	0.286648266057006+0.835765638441979i	3+2i
52	0.286648266057006+0.835765638441979i	0.28665462107676+0.835757784944302i	3+2i
53	0.28665462107676+0.835757784944302i	0.286647845749853+0.835753337822387i	3+2i
54	0.286647845749853+0.835753337822387i	0.286644835658166+0.835759100529263i	3+2i
55	0.286644835658166+0.835759100529263i	0.286649675566209+0.83576104416929i	3+2i

56	0.286649675566209+0.83576104416929i	0.286650841432823+0.835757025873764i	3+2i
57	0.286650841432823+0.835757025873764i	0.28664754095854+0.83575641538437i	3+2i
58	0.28664754095854+0.83575641538437i	0.286647316603915+0.835759098619603i	3+2i
59	0.286647316603915+0.835759098619603i	0.286649476363287+0.835759063917169i	3+2i
60	0.286649476363287+0.835759063917169i	0.286649276579844+0.835757342664514i	3+2i
61	0.286649276579844+0.835757342664514i	0.286647918514995+0.835757639282649i	3+2i
62	0.286647918514995+0.835757639282649i	0.286648263494781+0.835758699719912i	3+2i
63	0.286648263494781+0.835758699719912i	0.286649082497832+0.835758339848965i	3+2i
64	0.286649082497832+0.835758339848965i	0.286648729977283+0.835757714756507i	3+2i
65	0.286648729977283+0.835757714756507i	0.28664825908711+0.835758045959497i	3+2i
66	0.28664825908711+0.835758045959497i	0.286648560987457+0.835758395456791i	3+2i
67	0.286648560987457+0.835758395456791i	0.286648815917922+0.835758126619345i	3+2i
68	0.286648815917922+0.835758126619345i	0.286648581007636+0.835757944542097i	3+2i
69	0.286648581007636+0.835757944542097i	0.286648454381935+0.835758146565533i	3+2i
70	0.286648454381935+0.835758146565533i	0.28664862573543+0.835758231547522i	3+2i
71	0.28664862573543+0.835758231547522i	0.286648679919144+0.835758087994228i	3+2i
72	0.286648679919144+0.835758087994228i	0.286648561011147+0.83575805618008i	3+2i
73	0.286648561011147+0.83575805618008i	0.286648545089772+0.835758153632757i	3+2i
74	0.286648545089772+0.835758153632757i	0.286648624149542+0.835758158577259i	3+2i
75	0.286648624149542+0.835758158577259i	0.286648621797283+0.835758095074154i	3+2i
76	0.286648621797283+0.835758095074154i	0.286648571293602+0.835758102011527i	3+2i
77	0.286648571293602+0.835758102011527i	0.286648580855276+0.835758141773147i	3+2i
78	0.286648580855276+0.835758141773147i	0.286648611832995+0.835758130972521i	3+2i
79	0.286648611832995+0.835758130972521i	0.286648600743245+0.835758107105229i	3+2i
80	0.286648600743245+0.835758107105229i	0.286648582574866+0.835758117859238i	3+2i
81	0.286648582574866+0.835758117859238i	0.2866485926068+0.835758131505235i	3+2i
82	0.2866485926068+0.835758131505235i	0.286648602700354+0.835758122410018i	3+2i
83	0.286648602700354+0.835758122410018i	0.286648594635918+0.835758115077553i	3+2i
84	0.286648594635918+0.835758115077553i	0.28664858942535+0.835758122099179i	3+2i
85	0.28664858942535+0.835758122099179i	0.286648595445494+0.835758125699039i	3+2i
86	0.286648595445494+0.835758125699039i	0.286648597839406+0.835758120606661i	3+2i
87	0.286648597839406+0.835758120606661i	0.286648593583696+0.835758119101468i	3+2i
88	0.286648593583696+0.835758119101468i	0.286648592721264+0.835758122618506i	3+2i
89	0.286648592721264+0.835758122618506i	0.286648595597439+0.835758123026711i	3+2i
90	0.286648595597439+0.835758123026711i	0.286648595694123+0.83575812069829i	3+2i
91	0.286648595694123+0.83575812069829i	0.286648593827769+0.835758120806636i	3+2i
92	0.286648593827769+0.835758120806636i	0.286648594062958+0.83575812228781i	3+2i
93	0.286648594062958+0.83575812228781i	0.28664859522656+0.835758121982057i	3+2i
94	0.28664859522656+0.835758121982057i	0.286648594889782+0.835758121077576i	3+2i
95	0.286648594889782+0.835758121077576i	0.286648594194619+0.835758121418477i	3+2i
96	0.286648594194619+0.835758121418477i	0.286648594522129+0.835758121946225i	3+2i
97	0.286648594522129+0.835758121946225i	0.286648594917308+0.835758121642745i	3+2i
98	0.286648594917308+0.835758121642745i	0.286648594643569+0.835758121351471i	3+2i
99	0.286648594643569+0.835758121351471i	0.286648594432872+0.835758121593192i	3+2i
100	0.286648594432872+0.835758121593192i	0.286648594642613+0.835758121742124i	3+2i
101	0.286648594642613+0.835758121742124i	0.286648594744786+0.835758121562832i	3+2i
102	0.286648594744786+0.835758121562832i	0.286648594593525+0.835758121495557i	3+2i
103	0.286648594593525+0.835758121495557i	0.286648594551874+0.835758121621657i	3+2i
104	0.286648594551874+0.835758121621657i	0.286648594655853+0.835758121644862i	3+2i
105	0.286648594655853+0.835758121644862i	0.286648594666091+0.835758121560011i	3+2i
106	0.286648594666091+0.835758121560011i	0.286648594597542+0.835758121558597i	3+2i

107	0.286648594597542+0.835758121558597i	0.286648594601876+0.83575812161343i	3+2i
108	0.286648594601876+0.83575812161343i	0.2866485946453+0.835758121605599i	3+2i
109	0.2866485946453+0.835758121605599i	0.28664859463559+0.835758121571562i	3+2i
110	0.28664859463559+0.835758121571562i	0.286648594609195+0.835758121582027i	3+2i
111	0.286648594609195+0.835758121582027i	0.286648594619652+0.835758121602263i	3+2i
112	0.286648594619652+0.835758121602263i	0.28664859463497+0.835758121592299i	3+2i
113	0.28664859463497+0.835758121592299i	0.286648594625798+0.835758121580871i	3+2i
114	0.286648594625798+0.835758121580871i	0.286648594617406+0.835758121589101i	3+2i
115	0.286648594617406+0.835758121589101i	0.286648594624642+0.83575812159514i	3+2i
116	0.286648594624642+0.83575812159514i	0.286648594628889+0.835758121588889i	3+2i
117	0.286648594628889+0.835758121588889i	0.286648594623558+0.835758121585991i	3+2i
118	0.286648594623558+0.835758121585991i	0.286648594621672+0.835758121590482i	3+2i
119	0.286648594621672+0.835758121590482i	0.286648594625404+0.835758121591625i	3+2i
120	0.286648594625404+0.835758121591625i	0.286648594626021+0.835758121588558i	3+2i
121	0.286648594626021+0.835758121588558i	0.286648594623523+0.835758121588308i	3+2i
122	0.286648594623523+0.835758121588308i	0.286648594623521+0.83575812159032i	3+2i
123	0.286648594623521+0.83575812159032i	0.286648594625131+0.835758121590167i	3+2i
124	0.286648594625131+0.835758121590167i	0.28664859462488+0.835758121588894i	3+2i
125	0.28664859462488+0.835758121588894i	0.286648594623884+0.835758121589195i	3+2i
126	0.286648594623884+0.835758121589195i	0.286648594624201+0.835758121589962i	3+2i
127	0.286648594624201+0.835758121589962i	0.28664859462479+0.835758121589653i	3+2i
128	0.28664859462479+0.835758121589653i	0.286648594624495+0.835758121589204i	3+2i
129	0.286648594624495+0.835758121589204i	0.28664859462416+0.835758121589474i	3+2i
130	0.28664859462416+0.835758121589474i	0.286648594624402+0.835758121589722i	3+2i
131	0.286648594624402+0.835758121589722i	0.286648594624581+0.835758121589507i	3+2i
132	0.286648594624581+0.835758121589507i	0.286648594624395+0.835758121589382i	3+2i
133	0.286648594624395+0.835758121589382i	0.286648594624312+0.835758121589545i	3+2i
134	0.286648594624312+0.835758121589545i	0.286648594624446+0.835758121589593i	3+2i
135	0.286648594624446+0.835758121589593i	0.286648594624476+0.835758121589485i	3+2i
136	0.286648594624476+0.835758121589485i	0.286648594624387+0.835758121589471i	3+2i
137	0.286648594624387+0.835758121589471i	0.286648594624384+0.835758121589543i	3+2i
138	0.286648594624384+0.835758121589543i	0.286648594624441+0.83575812158954i	3+2i
139	0.286648594624441+0.83575812158954i	0.286648594624432+0.83575812158949i	3+2i
140	0.286648594624432+0.83575812158949i	0.286648594624395+0.835758121589507i	3+2i
141	0.286648594624395+0.835758121589507i	0.28664859462441+0.835758121589531i	3+2i
142	0.28664859462441+0.835758121589531i	0.286648594624427+0.835758121589515i	3+2i
143	0.286648594624427+0.835758121589515i	0.286648594624414+0.835758121589506i	3+2i
144	0.286648594624414+0.835758121589506i	0.286648594624408+0.835758121589516i	3+2i
145	0.286648594624408+0.835758121589516i	0.286648594624416+0.83575812158952i	3+2i
146	0.286648594624416+0.83575812158952i	0.286648594624418+0.835758121589512i	3+2i
147	0.286648594624418+0.835758121589512i	0.286648594624413+0.835758121589514i	3+2i
148	0.286648594624413+0.835758121589514i	0.286648594624413+0.835758121589514i	3+2i
149	0.286648594624413+0.835758121589514i	0.286648594624413+0.835758121589514i	3+2i
150	0.286648594624413+0.835758121589514i	0.286648594624413+0.835758121589514i	3+2i
151	0.286648594624413+0.835758121589514i	0.286648594624413+0.835758121589514i	3+2i
152	0.286648594624413+0.835758121589514i	0.286648594624413+0.835758121589514i	3+2i
153	0.286648594624413+0.835758121589514i	0.286648594624413+0.835758121589514i	3+2i
154	0.286648594624413+0.835758121589514i	0.286648594624413+0.835758121589514i	3+2i
155	0.286648594624413+0.835758121589514i	0.286648594624413+0.835758121589514i	3+2i
156	0.286648594624413+0.835758121589514i	0.286648594624413+0.835758121589514i	3+2i

Appendix 1; use of the Lambert Function:

Consider a classic problem;

$$x^{y} = y^{x}$$

Solve parametrically;

$$Let\ y = tx \qquad x^{tx} = (tx)^{x}$$

$$x^{(tx)\left(\frac{1}{x}\right)} = (tx)^{x\left(\frac{1}{x}\right)} \qquad x^{t} = tx \qquad x^{t-1} = t$$

$$x = t^{\left(\frac{1}{t-1}\right)} \qquad y = tx = t^{\left(\frac{1}{t-1}\right)+1} = t^{\left(\frac{1}{t-1}\right)+\left(\frac{t-1}{t-1}\right)} = t^{\left(\frac{t}{t-1}\right)}$$

$$\boxed{x = t^{\left(\frac{1}{t-1}\right)} \qquad y = t^{\left(\frac{t}{t-1}\right)}}$$

t	y	x	y^x = x^y
2	4	2	16
2.5	4.605039373	1.842015749	16.66043253
3	5.196152423	1.732050808	17.36190525
10	12.91549665	1.291549665	27.23047288
1000	1006.938631	1.006938631	1056.427588
π	5.361547173	1.70663347	17.56430411

Use of the Lambert Function to solve;

$$x^{y} = y^{x} \qquad x^{y\left(\frac{1}{xy}\right)} = y^{x\left(\frac{1}{xy}\right)} \qquad x^{\left(\frac{1}{x}\right)} = y^{\left(\frac{1}{y}\right)}$$

$$\frac{\lg(x)}{x} = \frac{\lg(y)}{y} = \lg(y)\, e^{\lg(y)^{-1}} = \lg(y)\, e^{-\lg(y)} = \ln(y)\, e^{-\ln(y)}$$

$$\frac{-\ln(x)}{x} = -\ln(y)e^{-\ln(y)}$$

$$\omega\left(-\frac{\ln(x)}{x}\right) = -\ln(y)$$

$$\ln(y) = -\omega\left(-\frac{\ln(x)}{x}\right) \qquad y = e^{-\omega\left(-\frac{\ln(x)}{x}\right)}$$

$$e^{\omega(x)} = \frac{x}{\omega(x)} \qquad e^{-\omega(x)} = \frac{\omega(x)}{x} \qquad y = \frac{\omega\left(-\frac{\ln(x)}{x}\right)}{-\frac{\ln(x)}{x}}$$

Appendix 2; Further Infinite Tetration & the Eigenlog:

Consider the following infinite tetration identities:

$$If \quad \lim_{infinite\ number\ of\ x} x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{y}}}}}}}} = y \quad ,then \quad x^y = y$$

$$Let \quad x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{y}}}}}}}} \quad symbolize\ an\ infinite\ tetration$$

$$y = x^y = x^{x^y} = x^{x^{x^y}} = x^{x^{x^{x^y}}} = x^{x^{x^{x^{x^y}}}} = x^{x^{x^{x^{x^{x^y}}}}} = \cdots = x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{y}}}}}}}}$$

Furthermore, for any value of "q":

$$If \quad x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{q}}}}}}}} = y \quad ,then \quad x^y = y$$

$$x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{q}}}}}}}} = x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{y}}}}}}}}$$

It does not matter what value "q" has; the tetration converges to "y". Note:

$$If \quad y = x^y \quad ,then \quad lg_x y = y$$

The value "x" can be referred to as the eigenbase, and the value "y" can be referred to as the eigenlog. Note the following designation for the iterated logarithm operation:

$$y = \lim_{take\ lg\ \infty times} lg_x\left(lg_x \ldots \left(lg_x\left(lg_x\left(lg_x(lg_x(any\ number))\right)\right)\right)\right)$$

$$lg_x(y) = \frac{ln(y)}{ln(x)} \qquad lg_{a+bi}(c+di) = \frac{\ln\left(\sqrt{c^2+d^2}\right) + \arctan\left(\frac{d}{c}\right)i}{\ln\left(\sqrt{a^2+b^2}\right) + \arctan\left(\frac{b}{a}\right)i}$$

To find the value of the eigenbase "x" if you know the eigenlog "y":

$$y = x^y \qquad x = y^{\frac{1}{y}}$$

To find the value of the eigenlog “y” if you know the eigenbase “x”:

$$y = \lim_{take\ lg\ \infty times} lg_x\left(lg_x \ldots \left(lg_x\left(lg_x\left(lg_x\left(lg_x(any\ number)\right)\right)\right)\right)\right)$$

Note that the eigenlog can be multivalued for a given eigenbase. Consider the following tetrations of iterated logarithms:

$$e^{e^{e^{ln(ln(ln(7)))}}} = e^{e^{ln(ln(7))}} = e^{(ln\quad)} = 7$$

$$ln(ln(ln(7))) = -0.406871380578537$$

$$e^{e^{e^{-0.40687138057853}}} = 7$$

When this same iterative logarithm process for the natural log of “7” is taken 110 times, the value converges to:

$$ln(ln \ldots (ln(ln(7))))\ take\ 110\ times = 0.318131505204764 + 1.33723570143069i$$

When this value of the iterated logarithm is tetrated at the top of an “e” tower 110 e’s high, the value of the tetration does not change and remains at *0.318131505204763 + 1.33723570143069i*.

$$e^{e^{\cdot^{\cdot^{\cdot^{e^{0.318131505204764+1.33723570143069i}}}}}} = 0.318131505204764 + 1.33723570143069i$$

At some point, the iterated logarithm converges to the eigenlog. See the Table below for the convergence of the tetration tower for the iterated natural logarithm of “7”.

# of Iterations	Value of iterated ln	Value after tetration
3	-0.406871380578537	7
20	0.325510597151267+1.33427068671393i	≈ 7
60	0.318131483991095+1.33723571201177i	≈ 7
80	0.318131505224506+1.3372357014665i	7.00172911203762+0.000430354365317284i
90	0.318131505206403+1.33723570143113i	7.01301298221841+0.0210468281605552i
100	0.318131505204825+1.33723570143065i	7.28970819256402-1.02089439001684i
105	0.318131505204775+1.33723570143068i	7.44312430449915+5.82005870739645i
110	0.318131505204764+1.33723570143069i	0.318131505204763+1.33723570143069i
>110	0.318131505204764+1.33723570143069i	0.318131505204763+1.33723570143069i

Note that we can calculate an eigenbase from an eigenlog:

$$lg_x(y) = lg_e(y) = \ln(y) \qquad x = y^{\frac{1}{y}}$$

$$x = (0.318131505204763 + 1.33723570143069\mathrm{i})^{\frac{1}{0.31813150520476\quad .33723570143069\mathrm{i}}} = e$$

The ultimate convergence of an iterated logarithm on the eigenlog can be justified as follows:

$$x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{y}}}}}}}} = y \qquad tetration\ is\ infinite \qquad x^y = y \qquad lg_x(y) = y$$

$$lg_x(lg_x(lg_{x....}\left(x^{x^{\cdot^{\cdot^{\cdot^{x^y}}}}}\right))\ ...) = y$$

$$x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{q}}}}}}}} = y \qquad tetration\ is\ infinite \qquad x^y = y \qquad lg_x(y) = y$$

$$lg_x(lg_x(lg_{x....}\left(x^{x^{\cdot^{\cdot^{\cdot^{x^q}}}}}\right))\ ...) = y$$

It does not matter what number the iteration is started with; the infinite iteration converges to the eigenlog.

Appendix 3; The General Tetration Problem:

If “x” & “y” are known, then “z” is easily calculated from the equation as is. Remember to start at the highest point on the tetration tower and work down.

$$x^{x^{x^{x^{x^{x^{y}}}}}} = z \qquad x, y\ known; z\ sought$$

$$2^{2^{2^{2^{(1.4)}}}} = 3.81169146335503x10^{22}$$

$$e^{e^{e^{e^{(0.1)}}}} = 788762617.503219$$

$$2^{2^{2^{2}}} = 65536 \qquad (((2)^2)^2)^2 = 2^6 = 256$$

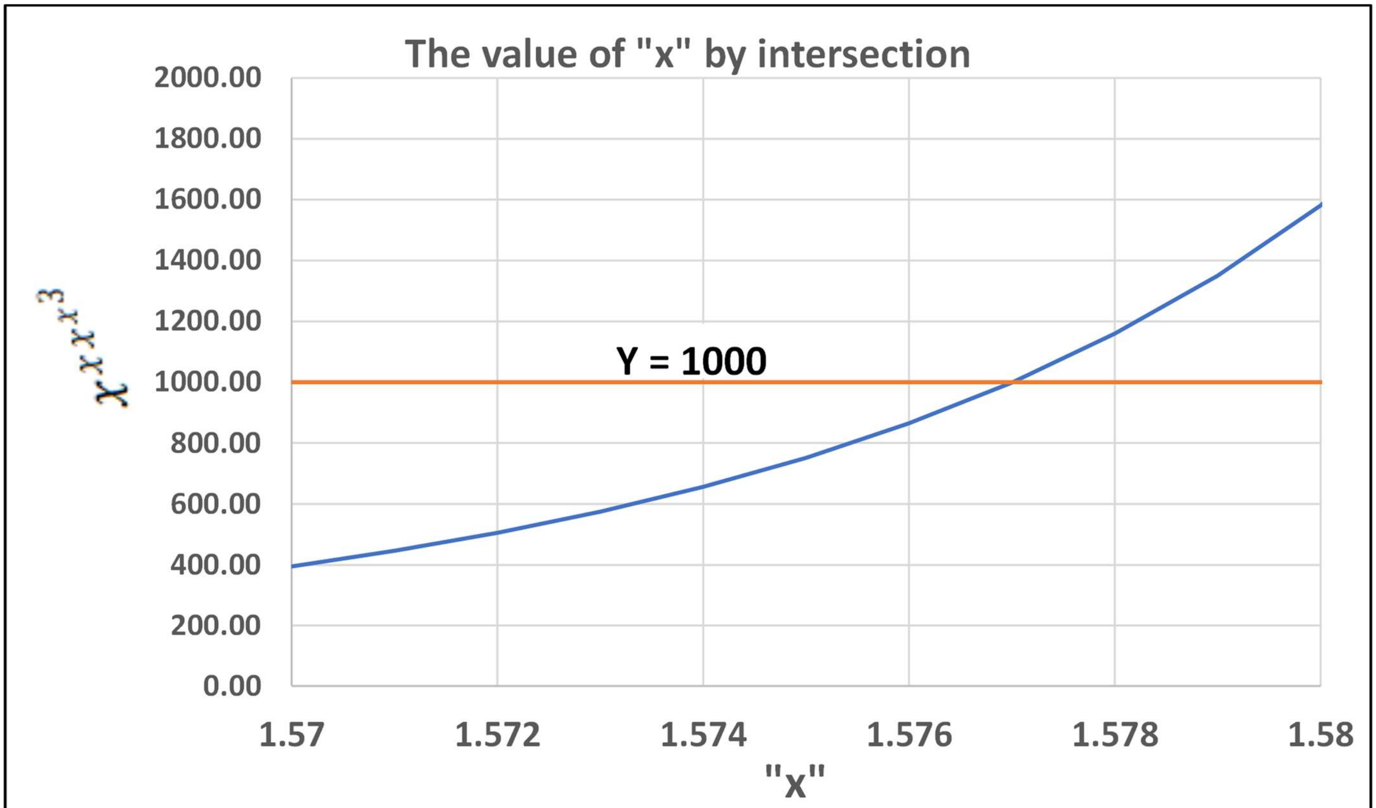
$$x^{x^{x^{x^{x^{x^{y}}}}}} = z \qquad y, z\ known; x\ sought$$

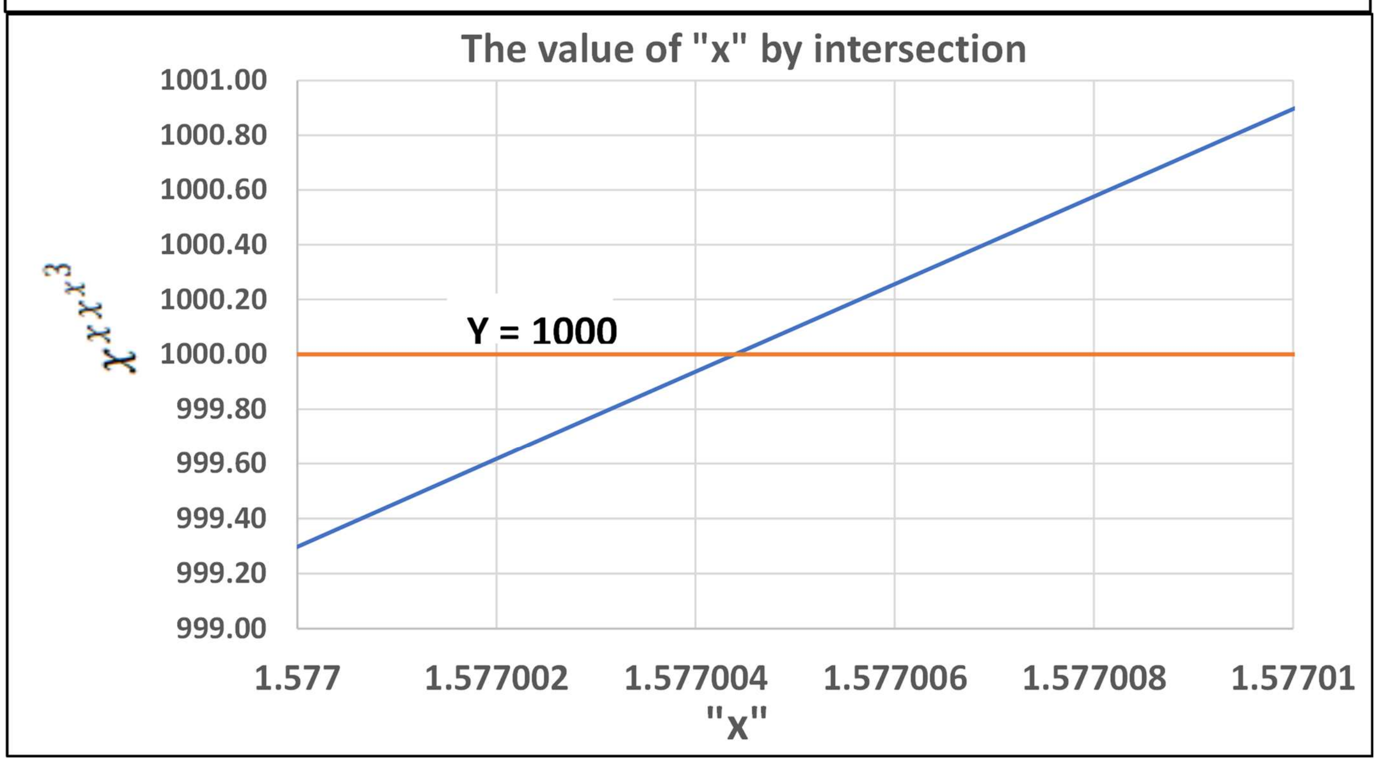
$$x^y = z \qquad x = z^{\frac{1}{y}}$$

$$x^{x^y} = z \qquad x = z^{\frac{\omega(yln(z))}{yln(z)}} = e^{\frac{\omega(yln(z))}{y}}$$

Further calculation can be done graphically.

$$x^{x^{x^{x^3}}} = 1000$$





$$x^{x^{x^{x^{x^{x^{y}}}}}} = z \qquad x, z\ known;\ y\ sought$$

$$lg_x(lg_x(lg_x(lg_x(lg_x(lg_x\left(x^{x^{x^{x^{x^{x^{y}}}}}}\right)))))) = y$$

$$y = lg_x(lg_x(lg_x(lg_x(lg_x(lg_x(z))))))$$

$$2^{2^{2^{2^{2^{2^{y}}}}}} = 1000$$

$$y = lg_2(lg_2(lg_2(lg_2(lg_2(lg_2(1000)))))) = -1.56123298948403 + 4.53236014182719i$$

$$3^{3^{3^{y}}} = 100$$

$$y = lg_3(lg_3(lg_3(100))) = 0.241954594316355$$

Recall that this method eventually breaks down, as the iterated logarithm converges on the value of the eigenlog "y" for the given eigenbase "x" (see previous discussion).

$$x^y = y \qquad lg_x(y) = y$$

Appendix 4; Multivalued Eigenlogs in the Tetration Problem:

Consider the two related processes of infinite tetration and the infinite logarithm:

$$\lim_{infinite}\left(x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{anything}}}}}}}\right) = y \qquad y = x^y \qquad x = y^{\frac{1}{y}}$$

$$lg_x(lg_x(lg_{x\ldots.}(anything)))\ldots) = y \qquad lg_x y = y \qquad x = y^{\frac{1}{y}}$$

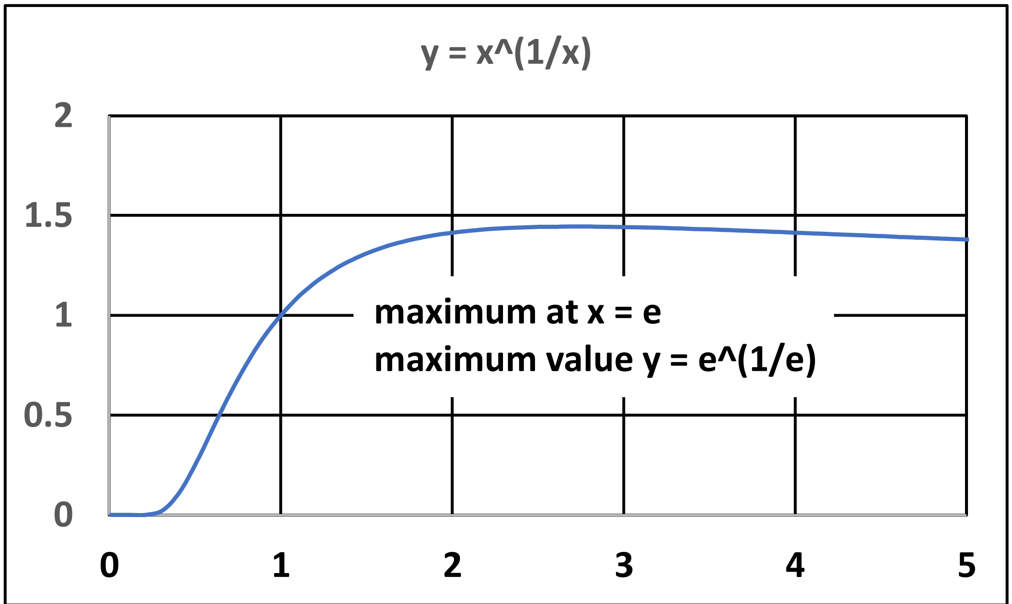
Consider the following tetration tower and associated infinite log operation:

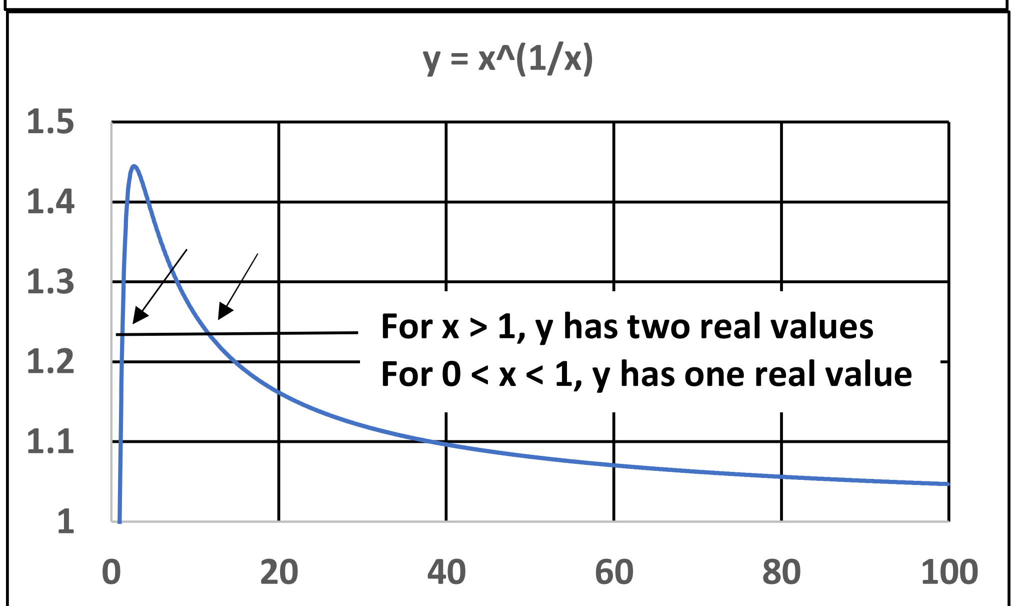
$$infinite\ tower:\quad 7^{7^{7^{\cdot^{\cdot^{\cdot^{7^{any\,real\,number}}}}}}} = \infty$$

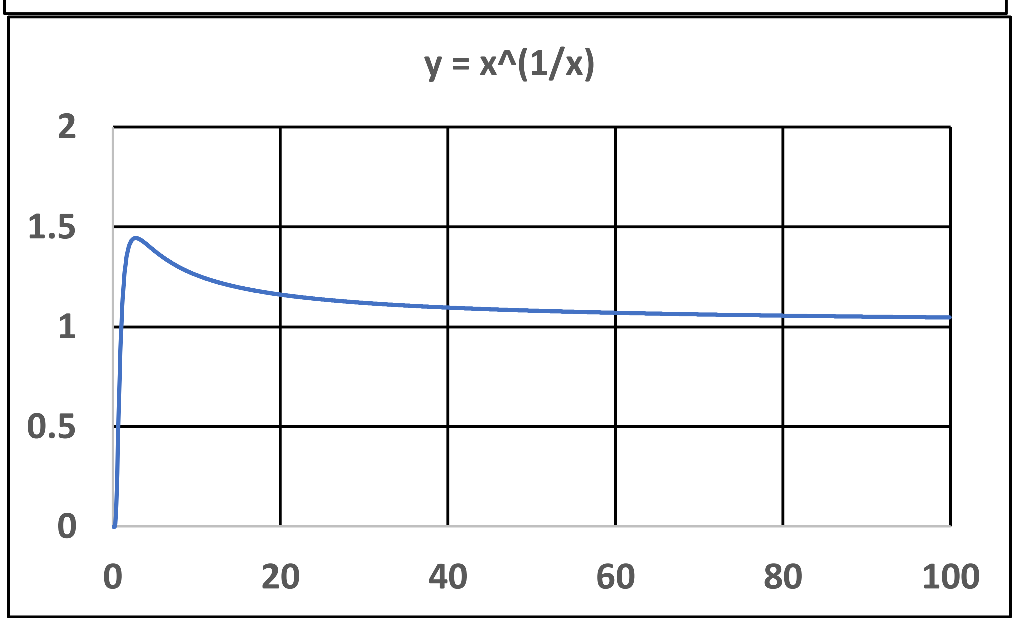
$$infinite:\ lg_7(lg_7(lg_{7\ldots.}(any\ \#)))\ldots) = -0.0786825549987237 + 0.854420793135892i$$

$$7^{7^{7^{\cdot^{\cdot^{\cdot^{7^{(-0.0786825549987237+0.85442079313589\ \ )}}}}}}} = -0.0786825549987237+0.8544207931358$$

Consider the graph below of the function y = x^(1/x) for real numbers greater than zero.







The value of "y" in these graphs is the eigenbase ("x" in the tetration tower above). The value of "x" in these graphs is the eigenlog ("y" in the tetration tower above). These graphs show that real values of "y" (eigenbase) map to two real values of "x" (eigenlog) for x > 1. Real values of "x" (eigenlog) less than or equal to one but greater than zero (0 < x ≤ 1) map to one real value of "y" (eigenbase). Consider an infinite tetration and the associated infinite logarithm for a value of the eigenbase of "x" = 1.3 (note that this value of "x" is "y" in the graphs above but is consistent with our use of "x" as the eigenbase in all the tetration formulas). The tetration and logarithm operations are carried out for a sufficient number of iterations such that the value of the result converges to a constant to 15 decimal places.

$$1.3^{1.3^{1.3^{1.3^{\cdot^{\cdot^{\cdot^{1.3^{4\ or\ any\ value\ \leq highe\ \ value\ of\ eigenlog}}}}}}}} = 1.47098896009016$$

$$1.47098896009016^{\left(\frac{1}{1.47098896009016}\right)} = 1.3 = eigenbase = eigenlog^{\frac{1}{eigenlog}}$$

$$lg_{1.3}(lg_{1.3}(lg_{1.3....}(2\ or\ any\ value)))\ ...) = 7.8570653511643$$

$$7.8570653511643^{\left(\frac{1}{7.8570653511643}\right)} = 1.3 = eigenbase = eigenlog^{\frac{1}{eigenlog}}$$

The Table below shows the infinite tetration and infinite logarithm operation for an eigenbase of 1.3 (x = 1.3).

Iteration	Tetration (x = 1.3)	Infinite Logarithm with Base = 1.3
0	4 (starting value at top of tower)	2 (starting value)
1	2.85609999999999	2.64192679581114
2	2.11560024442779	3.7028994776384
3	1.74204184962474	4.98968927046774
4	1.57940710805364	6.12649607841836
5	1.51343209792132	6.90880287368666
6	1.48746079113978	7.36684312250763
7	1.47735977135808	7.61151400288259
8	1.47344973471916	7.73604631459922
9	1.47193896562844	7.79790170425028
10	1.47135564611319	7.82825622002397
11	1.47113048384475	7.84306423894654
12	1.47104358006544	7.85026729530543
13	1.47101003999789	7.85376615887153
14	1.47099709558551	7.85546456331664
15	1.47099209986539	7.85628872328449
16	1.47099017183961	7.8566885871678
17	1.47098942774668	7.85688257701921
18	1.47098914057516	7.85697668564422
19	1.47098902974563	7.85702233891351
20	1.47098898697264	7.85704448568759
500	1.47098896009016	7.8570653511643
501	1.47098896009016	7.8570653511643

Note that the tetration operation converges on the lower value of the eigenlog while the infinite logarithm converges of the higher value of the eigenlog. Starting tetration values greater than 7.857 (the higher valued eigenlog) diverge to infinity.

Consider the same two infinite operations for a complex eigenbase.

$$(1+i)^{(1+i)^{(1+i)^{(1+i)^{\cdot^{\cdot^{\cdot^{(1+i)^{real\ value\ \leq 7.7\ \ or\ any\ complex\ value}}}}}}}} = 0.641026478820489 + 0.523628461257163\mathrm{i}$$

$$(0.641026478820489 + 0.523628461257163\mathrm{i})^{\left(\frac{1}{0.641026478820489+\ .523628461257163\mathrm{i}}\right)} = 1+\mathrm{i}$$

$$lg_{(1+i)}(lg_{(1+i)}(lg_{(1+i)\dots}(any\ value)))\ \dots) = -1.92703521293561 - 2.22489033258259\mathrm{i}$$

$$(-1.92703521293561 - 2.22489033258259\mathrm{i})^{\left(\frac{1}{-1.92703521293561-2.22489033258259\mathrm{i}}\right)} = 1+\mathrm{i}$$

The Table below shows the infinite tetration and infinite logarithm operation for the complex valued eigenbase of 1 + i (“x” = 1 + i).

Iteration	Tetration (x = 1 + i)	Infinite Logarithm with Base = 1 + i
0	4 (starting value at top of tower)	2 (starting value)
1	-4+1.29245572577652E-14i	0.32596797216346-0.73870212227295i
2	-0.249999999999998-8.07784828610318E-16i	-1.33174187217976-0.315307687385815i
3	0.899384067645391-0.17889861407889i	-2.95275039841678-1.70245381488511i
4	1.25660205880593+0.944164689257731i	-2.21401486607585-2.538276197507i
5	0.18691160740567+0.712239979506613i	-1.8673562268413-2.37020166747178i
6	0.563167212716624+0.2338949265717i	-1.86580270050531-2.22949712218545i
7	0.876137640077131+0.505573770704648i	-1.91477931556528-2.20373092792962i
8	0.591935330708955+0.692222167523271i	-1.9325629617224-2.21689889866666i
9	0.542982288748357+0.461831304725132i	-1.93082989142914-2.22552335565694i
10	0.699484994029026+0.464822624155441i	-1.92744950978143-2.22635460510356i
11	0.670557296128982+0.576910827228948i	-1.92658147152254-2.22528611848047i
12	0.599410368758997+0.532760058132547i	-1.92681686339407-2.22479491970198i
13	0.642183912239331+0.493702236342137i	-1.92703243201033-2.22479606704305i
14	0.661580221097189+0.530060246182742i	-1.92706847316511-2.2248733987807i
15	0.632608086600559+0.536426599882601i	-1.92704689586985-2.22489937048439i
16	0.633889924865351+0.515502944199075i	-1.92703395442266-2.22489604109727i
17	0.647940724063073+0.520241148939854i	-1.92703295753572-2.22489084748892i
18	0.642030212357509+0.529020565070375i	-1.92703464733905-2.22488961262397i
19	0.637151410183289+0.523291840505192i	-1.92703537491566-2.22489000846818i
20	0.641985183733904+0.521037816931915i	0.32596797216346-0.73870212227295i
500	0.641026478820489+0.523628461257163i	-1.92703521293561-2.22489033258259i
501	0.641026478820489+0.523628461257163i	-1.92703521293561-2.22489033258259i

The infinite tetration converges to one value of eigenlog while the infinite logarithm operation converges to another value of the eigenlog. Note that the starting value in the tetration tower must be equal to or less than ≈ 7.7 (determined empirically for now). Complex starting values for the tetration tower don’t appear to be limited. The starting value for the infinite logarithm can’t equal zero, one, or 1 + i (the value of the eigenbase) but otherwise does not appear to be limited.

These two operations (*i.e.*, infinite tetration and infinite logarithm) can display more complicated behavior. Consider these two operations for an eigenbase of 0.5 (“x” = 0.5).

$$(0.5)^{(0.5)^{(0.5)^{(0.5)^{\cdot^{\cdot^{\cdot^{(0.5)^{4\ or\ any\ real\ value}}}}}}}} = 0.641185744504986$$

$$(0.641185744504986)^{\left(\frac{1}{0.641185744504986}\right)} = 0.5$$

The infinite logarithm operation converges to an oscillating value for the eigenlog.

$$lg_{(0.5)}(lg_{(0.5)}(lg_{(0.5)\ldots}(2\ or\ any\ value)))\ldots) = -1.82915391805471 - 3.04631566959698\mathrm{i}$$

$$lg_{(0.5)}(lg_{(0.5)}(lg_{(0.5)\ldots}(2\ or\ any\ value)))\ldots) = -1.82915391805471 + 3.04631566959698\mathrm{i}$$

$$(-1.82915391805471 + 3.04631566959698\mathrm{i})^{\left(\frac{1}{-1.8291539180547\quad .04631566959698\mathrm{i}}\right)} = 0.5$$

$$(-1.82915391805471 - 3.04631566959698\mathrm{i})^{\left(\frac{1}{-1.82915391805471\quad .04631566959698\mathrm{i}}\right)} = 0.5$$

The Table below shows the infinite tetration and infinite logarithm operation for an eigenbase = 0.5 (“x” = 0.5).

Iteration	Tetration (x = 0.5)	Infinite Logarithm with Base = 0.5
0	4 (starting value at top of tower)	2 (starting value)
1	0.0625000000000001	-1
2	0.957603280698574	-4.53236014182719i
3	0.514911615151943	-2.18026250241721+2.2661800709136i
4	0.699835806020931	-1.65292083181801-3.37139671664877i
5	0.615642269297597	-1.90873299745176+2.92381761703653i
6	0.65263928223719	-1.80392974231071-3.10057230174413i
7	0.636115531774622	-1.84284177356933+3.0263846866889i
8	0.643443094327358	-1.82509908785243-3.05527118934705i
9	0.640183281243324	-1.83142567262132+3.04303761323921i
10	0.641631430183826	-1.82848828424475-3.04779226382715i
11	0.640987696296732	-1.82952883195404+3.04577548977293i
12	0.641273770318298	-1.82904426458086-3.04655909342017i
13	0.641146623850091	-1.8292157338582+3.04622662665302i
14	0.641203131371659	-1.82913584376107-3.04635579819056i
15	0.641178017200491	-1.82916410869035+3.04630099103011i
16	0.641189178807188	-1.8291509385655-3.04632228479706i
17	0.641184218178957	-1.8291555979883+3.0463132498409i
18	0.641186422859719	-1.82915342688774-3.0463167601122i
19	0.641185443019731	-1.82915419499183+3.04631527070055i
20	0.641185878495928	-1.82915383708593-3.04631584936831i
500	0.641185744504986	**-1.82915391805471 + 3.04631566959698i**
501	0.641185744504986	**-1.82915391805471 − 3.04631566959698i**

Note that all real valued eigenbases "x" with value "≈ 0.08 < x < 1" display the same behavior as shown above for the eigenbase of 0.5 ("x" = 0.5).

These two operations (*i.e.*, infinite tetration and infinite logarithm) display yet more complicated behavior for real values of "x" less than ≈ 0.08. For these values of the eigenbase, both the infinite tetration and the infinite logarithm converge to oscillation between two values; albeit close values for the eigenlog.

$$(0.05)^{(0.05)^{(0.05)^{(0.05)^{\cdot^{\cdot^{\cdot^{(0.05)^{4\ or\ any\ real\ value}}}}}}}} = 0.137359395737951$$

$$(0.05)^{(0.05)^{(0.05)^{(0.05)^{\cdot^{\cdot^{\cdot^{(0.05)^{4\ or\ any\ real\ value}}}}}}}} = 0.662660838900559$$

$$(0.137359395737951)^{\left(\frac{1}{0.662660838900559}\right)} = 0.05$$

$$(0.662660838900559)^{\left(\frac{1}{0.137359395737951}\right)} = 0.05$$

$$lg_{(0.05)}(lg_{(0.05)}(lg_{(0.05)\ldots}(any\ value)))\ \ldots) = 0.350224852743221$$

$$lg_{(0.05)}(lg_{(0.05)}(lg_{(0.05)\ldots}(any\ value)))\ \ldots) = 0.350224852743167$$

$$0.350224852743221^{\left(\frac{1}{0.350224852743167}\right)} = 0.05 = 0.350224852743167^{\left(\frac{1}{0.350224852743221}\right)}$$

The Table below shows the infinite tetration and infinite logarithm operation for an eigenbase of 0.5 (x = 0.5).

Iteration	Tetration (x = 0.05)	Infinite Logarithm with Base = 0.05
0	4 (starting value at top of tower)	2 (starting value)
1	6.24999999999977E-06	-0.231378213159759
2	0.999981276848571	0.488595604443032-1.04868939101249i
3	0.0500028045561025	-0.0486577165260118+0.378802249899551i
4	0.860884426409328	0.321309889463502-0.566989339629828i
5	0.07585138906842	0.142925374976128+0.352243109867889i
6	0.796737149688209	0.322868621869627-0.395674957485358i
7	0.0919223396310438	0.224317548654519+0.295880377748795i
8	0.759287550064707	0.330718935051145-0.307807385147343i
9	0.102835750527958	0.265218169770571+0.250199805329658i
10	0.734865068619474	0.336787892096599-0.252448520946769i
11	0.11064161412267	0.288864708432152+0.214715600442086i
12	0.717880144003636	0.341105626492605-0.213371801484955i
13	0.116416998871167	0.303915077565389+0.186592223500548i
14	0.705566564934875	0.344180248513929-0.18380222895423i
500	**0.137359395737951**	**0.350224852742416+1.15747720158856E-11i**
501	**0.662660838900559**	**0.350224852743935-1.10322091360067E-11i**
15000	**0.137359395737951**	**0.350224852743167**
15001	**0.662660838900559**	**0.350224852743221**

It is possible that some of this oscillatory behavior is caused because the limited accuracy (15 decimal places) of Excel is preventing convergence. That is, the limited accuracy of Excel prevents convergence beyond a certain point at which oscillatory values occur. More complicated oscillatory behavior is shown by larger complex eigenbases. The Table below shows the infinite tetration and infinite logarithm operation for an eigenbase of 3 + 2i ("x" = 3 + 2i).

The infinite logarithm converges to a single value.

$$lg_{(3+2i)}(lg_{(3+2i)}(lg_{(3+2i)....}(any\ value)))\ ...) = -0.36452405787166 - 1.27484729025114\mathrm{i}$$

$$(-0.36452405787166 - 1.27484729025114\mathrm{i})^{\left(\frac{1}{-0.36452405787166-\ .27484729025114\mathrm{i}}\right)} = 3 + 2\mathrm{i}$$

The infinite tetration converges to an oscillation between 16 values. Again, this may be a result of numerical problems in Excel. Some infinite tetrations diverge to ∞/–∞, and when the value of the tetration becomes a negative number of sufficiently high absolute value, then the next step in the tetration will generate zero, as the 15 decimal places of Excel are insufficient for keeping a miniscule number smaller than 4E-16. Once zero is entered into an Excel cell, the next step will generate one (any number raised to the zero power is one) and the next step after one is entered into an Excel cell will regenerate the eigenbase. Thus, a repeating pattern is generated whereas the actual result should be convergence. The inability of 15 decimal place numbers to keep track of very large and very small numbers may sometimes (possibly always) cause oscillation in the values converged upon.

Iteration	Tetration (x = 3 + 2i)	Infinite Logarithm with Base = 3 + 2i
0	4 (starting value at top of tower)	2 (starting value)
1	-119.000000000001+120.000000000001i	0.446595782252787-0.204759974643624i
2	-7.42738014308652E-98+9.32802392965206E-98i	-0.58489981331754-0.0670341775315805i
3	1+7.59563563070108E-98i	-1.23568160273274-1.79410837523042i
4	3+2i	-0.140523799209821-1.63067518920793i
5	-5.40973879391767-13.4104423704128i	-0.171973748618434-1.21299717847489i
6	0.105358669816736-2.57715749695565i	-0.374805080551977-1.16278882563838i
7	-5.18269622853561+0.528357221167636i	-0.427117518390044-1.27212477615303i
8	-0.000682007137239304-0.000663651712670805i	-0.370225772279565-1.30764978029219i
9	0.999514905821318-0.00125153173831102i	-0.347863915497787-1.28045707451794i
10	3.00411702449891+1.99455248004687i	-0.360162809545055-1.2665270903913i
11	-5.51751523913687-13.4996395140563i	-0.368581181146283-1.27185855093874i
12	-0.323088102618349-2.34514778337802i	-0.366428382970541-1.27675497409435i
13	-2.61963024680396+0.146791454525529i	-0.363669659741691-1.27600596574985i
14	0.00691579514873327-0.0311160383510963i	-0.363842952762337-1.27448960726217i
15	1.02687816124314-0.0368180776823752i	-0.364658110524776-1.27445842123545i
16	3.23805065355357+2.01473974004952i	-0.364740385252326-1.27488680457548i
17	-4.33182409506491-18.9654464469438i	-0.364520607795021-1.2749647886425i
18	-44.5962107021002-265.704983051494i	-0.364461690870018-1.27485493999037i
19	-8.60293446632849E+42-5.68246654921547E+42i	-0.364514986743545-1.27481494942746i
20	0	-0.364540417181449-1.27483988932287i
500	-2.61963024680396+0.146791454525529i	-0.36452405787166-1.27484729025114i
501	0.00691579514873327-0.0311160383510963i	-0.36452405787166-1.27484729025114i
15000	-5.40973879391767-13.4104423704128i	-0.36452405787166-1.27484729025114i
15001	0.105358669816736-2.57715749695565i	-0.36452405787166-1.27484729025114i

The infinite tetration for eigenbase = 3 + 2i displays convergence of the eigenlog to an oscillation between 16 values (see below).

Iteration	Tetration (x = 3 + 2i)
14979	0
14980	1
14981	3+2i
14982	-5.40973879391767-13.4104423704128i
14983	0.105358669816736-2.57715749695565i
14984	-5.18269622853561+0.528357221167636i
14985	-0.000682007137239304-0.000663651712670805i
14986	0.999514905821318-0.00125153173831102i
14987	3.00411702449891+1.99455248004687i
14988	-5.51751523913687-13.4996395140563i
14989	-0.323088102618349-2.34514778337802i
14990	-2.61963024680396+0.146791454525529i
14991	0.00691579514873327-0.0311160383510963i
14992	1.02687816124314-0.0368180776823752i
14993	3.23805065355357+2.01473974004952i
14994	-4.33182409506491-18.9654464469438i
14995	-44.5962107021002-265.704983051494i
14996	-8.60293446632849E+42-5.68246654921547E+42i
14997	0
14998	1
14999	3+2i
15000	-5.40973879391767-13.4104423704128i
15001	0.105358669816736-2.57715749695565i

Direct Calculation of the Eigenlog from the Eigenbase or the Eigenbase from the Eigenlog:

The condition for an eigenlog to a given eigenbase:

$$lg_x(y) = y$$

Use the convention that the angle theta of a complex number is to be reduced to a value between -π & π by sequential addition or subtraction of 2π. If the eigenlog (c & d) is known and the eigenbase (a & b) is sought:

$$lg_x(y) = y \qquad\qquad x = y^{\frac{1}{y}}$$

$$lg_{a+bi}(c+di) = c+di$$

$$a+bi = (c+di)^{\left(\frac{1}{c+di}\right)}$$

$$ln(a+bi) = \frac{1}{c+di} ln(c+di) = \frac{1}{c+di}\left(ln\left(\sqrt{c^2+d^2}\right) + arctan\left(\frac{d}{c}\right)i\right)$$

$$\frac{ln\left(\sqrt{c^2+d^2}\right)+arctan\left(\frac{d}{c}\right)i}{c+di}=\frac{\left(ln\left(\sqrt{c^2+d^2}\right)+arctan\left(\frac{d}{c}\right)i\right)(c-di)}{c^2+d^2}$$

$$ln(a+bi)=\frac{c\left(ln\left(\sqrt{c^2+d^2}\right)\right)+d\left(arctan\left(\frac{d}{c}\right)\right)}{c^2+d^2}+\frac{c\left(arctan\left(\frac{d}{c}\right)\right)-d\left(ln\left(\sqrt{c^2+d^2}\right)\right)}{c^2+d^2}i$$

$$a+bi=e^{\left(\frac{c\left(ln\left(\sqrt{c^2+d^2}\right)\right)+d\left(arctan\left(\frac{d}{c}\right)\right)}{c^2+d^2}+\frac{c\left(arctan\left(\frac{d}{c}\right)\right)-d\left(ln\left(\sqrt{c^2+d^2}\right)\right)}{c^2+d^2}i\right)}$$

$$a+bi=\sqrt{a^2+b^2}\left(e^{arctan\left(\frac{b}{a}\right)i}\right)$$

$$\sqrt{a^2+b^2}\left(e^{arctan\left(\frac{b}{a}\right)i}\right)=e^{\left(\frac{c\left(ln\left(\sqrt{c^2+d^2}\right)\right)+d\left(arctan\left(\frac{d}{c}\right)\right)}{c^2+d^2}\right)}e^{\left(\frac{c\left(arctan\left(\frac{d}{c}\right)\right)-d\left(ln\left(\sqrt{c^2+d^2}\right)\right)}{c^2+d^2}\right)i}$$

$$\sqrt{a^2+b^2}=e^{\left(\frac{c\left(ln\left(\sqrt{c^2+d^2}\right)\right)+d\left(arctan\left(\frac{d}{c}\right)\right)}{c^2+d^2}\right)}=p$$

$$arctan\left(\frac{b}{a}\right)=\frac{c\left(arctan\left(\frac{d}{c}\right)\right)-d\left(ln\left(\sqrt{c^2+d^2}\right)\right)}{c^2+d^2}=q$$

$$\frac{b}{a}=tan\ (q)$$

$$a^2+b^2=p^2 \qquad\qquad b=atan(q)$$

$$a^2+\left(atan(q)\right)^2=p^2$$

$$a = \sqrt{\frac{p^2}{\left(tan(q)\right)^2 + 1}} \qquad b = \sqrt{p^2 - \frac{p^2}{\left(tan(q)\right)^2 + 1}} = \sqrt{p^2\left(\frac{\left(tan(q)\right)^2}{\left(tan(q)\right)^2 + 1}\right)}$$

The value of "a" & "b" can be determined explicitly.

If the eigenbase (a & b) is known and the eigenlog (c & d) is sought:

$$lg_x(y) = \frac{ln(y)}{ln(x)} \qquad lg_{a+bi}(c+di) = \frac{ln\left(\sqrt{c^2+d^2}\right) + arctan\left(\frac{d}{c}\right)i}{ln\left(\sqrt{a^2+b^2}\right) + arctan\left(\frac{b}{a}\right)i}$$

$$lg_{a+bi}(c+di) = \frac{ln\left(\sqrt{c^2+d^2}\right) + arctan\left(\frac{d}{c}\right)i}{ln\left(\sqrt{a^2+b^2}\right) + arctan\left(\frac{b}{a}\right)i} = c + di$$

$$\frac{\left(ln\left(\sqrt{c^2+d^2}\right) + arctan\left(\frac{d}{c}\right)i\right)\left(ln\left(\sqrt{a^2+b^2}\right) - arctan\left(\frac{b}{a}\right)i\right)}{\left(ln\left(\sqrt{a^2+b^2}\right) + arctan\left(\frac{b}{a}\right)i\right)\left(ln\left(\sqrt{a^2+b^2}\right) - arctan\left(\frac{b}{a}\right)i\right)} = c + di$$

$$Let{:} \qquad arctan\left(\frac{b}{a}\right)i = \theta_x$$

$$arctan\left(\frac{d}{c}\right)i = \theta_y$$

$$\sqrt{a^2+b^2} = l_x$$

$$\sqrt{c^2+d^2} = l_y$$

$$\frac{\left(ln(l_y) + \theta_y i\right)\left(ln(l_x) - \theta_x i\right)}{\left(ln(l_x) + \theta_x i\right)\left(ln(l_x) - \theta_x i\right)} = \frac{\left(ln(l_y)ln(l_x) + \theta_x\theta_y\right) - \left(\theta_x ln\,(l_y) + \theta_x ln\,(l_y)\right)i}{\left(\left(ln(l_x)\right)^2 + (\theta_x)^2\right)}$$

$$\frac{\left(ln(l_y)ln(l_x) + \theta_x\theta_y\right)}{\left(ln(l_x)\right)^2 + (\theta_x)^2} = c \qquad \frac{\left(\theta_x ln\,(l_y) + \theta_y ln\,(l_x)\right)}{\left(ln(l_x)\right)^2 + (\theta_x)^2} = d$$

$$\left(\ln(l_x)\right)^2 + (\theta_x)^2 = f$$

$$\ln(l_x) = g \qquad \theta_x = h$$

$$\frac{g\left(ln\left(\sqrt{c^2+d^2}\right)\right)+h\left(\arctan\left(\frac{d}{c}\right)\right)}{f}=c$$

$$\frac{h\left(ln\left(\sqrt{c^2+d^2}\right)\right)+g\left(\arctan\left(\frac{d}{c}\right)\right)}{f}=d$$

$$\frac{g}{f}ln\left(\sqrt{c^2+d^2}\right)+\frac{h}{f}\arctan\left(\frac{d}{c}\right)=c$$

$$\frac{h}{f}ln\left(\sqrt{c^2+d^2}\right)+\frac{g}{f}\arctan\left(\frac{d}{c}\right)=d$$

There is an interesting symmetry in the formulas for "c" & "d", but a complete explicit solution for d & c via this method evades me for the moment. Alternatively (if you are able to use the Lambert Function with a complex number);

$$x=y^{\frac{1}{y}} \qquad\qquad y=e^{-\omega(-ln(x))}$$