Triangles; the beginning of logic and reason:

By: Michael D. Gernon, Ph.D.

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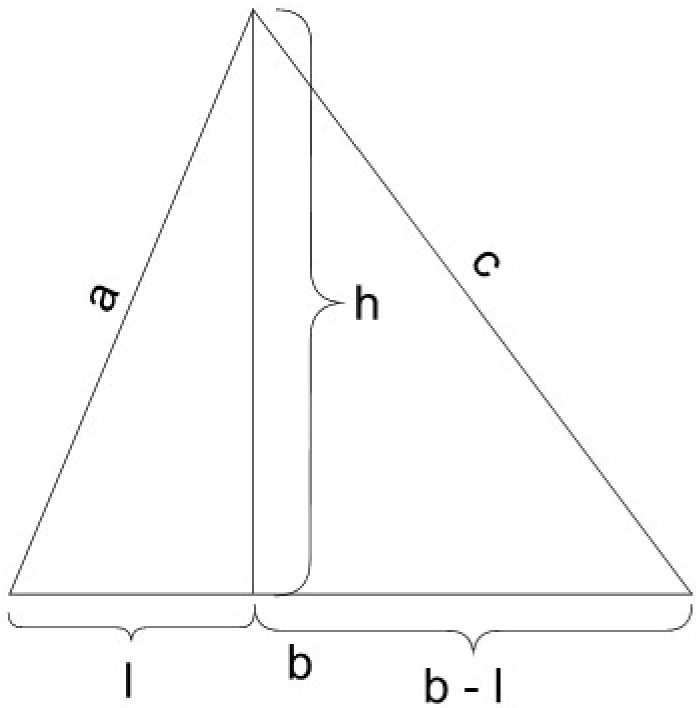
* Citations are made in the text. Commonly known ideas due to Euclid and Pythagoras are not cited.

1) The Area of a Triangle calculated form the side lengths:

The area of a triangle is equal to one half the base multiplied by the height. In order to calculate the area of a triangle given only the length of the sides, use the Pythagorean theorem as shown below:



$$h^2 + l^2 = a^2 \qquad h^2 + (b-l)^2 = c^2 \qquad h^2 + b^2 + l^2 - 2bl = c^2$$

$$a^2 + b^2 - 2bl = c^2 \qquad l = \frac{a^2 + b^2 - c^2}{2b}$$

$$h^2 + l^2 = a^2 \qquad h = \sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2}$$

$$Area = A = \sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2}\left(\frac{b}{2}\right)$$

For a 13, 14, 15 triangle, the magnitude of h, l, and the area (A) are:

$$l = \frac{13^2 + 14^2 - 15^2}{2(14)} = \frac{169 + 196 - 225}{28} = 5$$

$$h = \sqrt{13^2 - (5)^2} = \sqrt{144} = 12$$

$$A = \frac{hb}{2} = \frac{(12)(14)}{2} = 84$$

The area formula above is not as compact as Heron's formula, but it does show that the area of a triangle can be calculated solely from the lengths of the three sides. This formula can be converted via some algebra to Heron's formula.

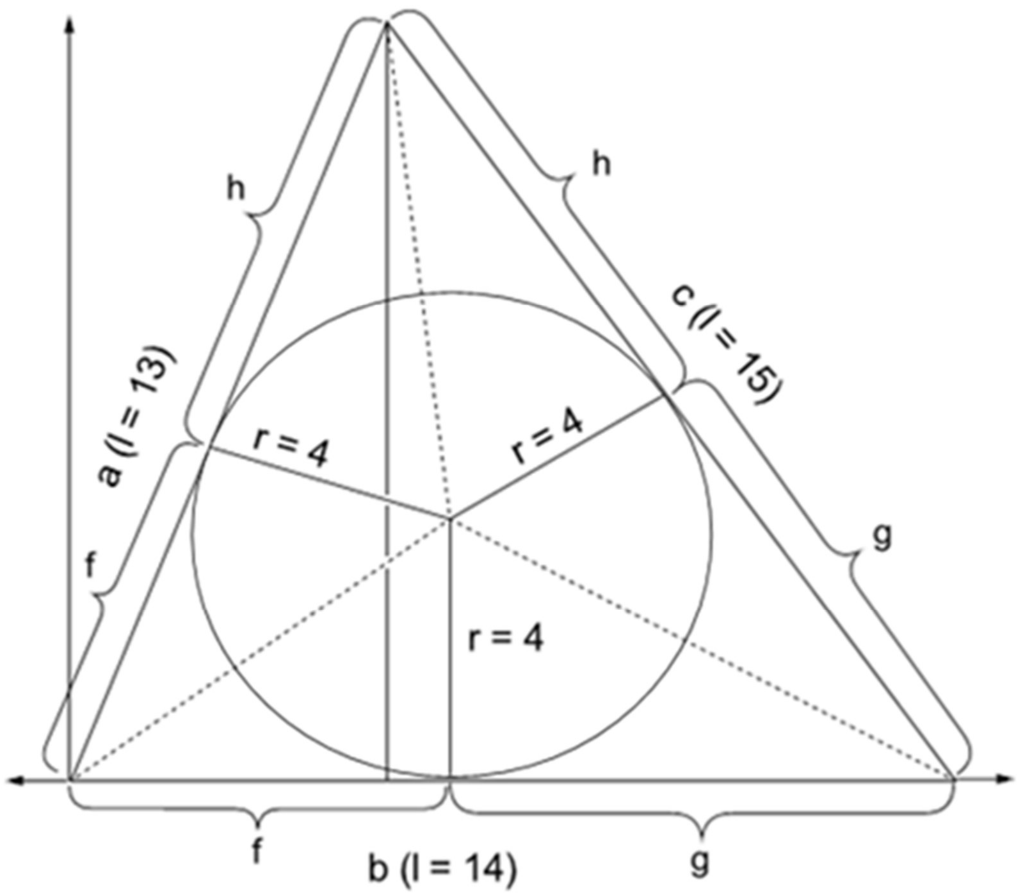
$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{a+b+c}{2}$$

$$A = \sqrt{21(6)(7)(8)} = \sqrt{3^2 7^2 2^4} = 84 \qquad s = \frac{13+14+15}{2} = 21$$

Given access to computers for fast calculations, one can use either form of the equation. Heron's formula is easier for calculation by hand.

<u>2) The radius of the incircle:</u>

To calculate the radius of the triangle's incircle, consider the following diagram:



The center of the incircle is equidistant from some point on each of the three sides of the triangle wherein a line from the center to a given side forms a right angle with the side. The dotted lines shown are always equidistant from two sides of the triangle (*i.e.*, the two sides converging at the vertex the dotted line originates from) wherein a line from a given point on the equidistant line to one of the two sides forms a right angle with side. We can conclude that the equidistant lines from two vertices intersect at some point which is equidistant from some point on each of the three sides. Logic dictates that the third equidistant line must also intersect at this same point. Further, one can conclude that the equidistant lines must bisect the angles they originate from. It is apparent that the area (A) of the triangle is:

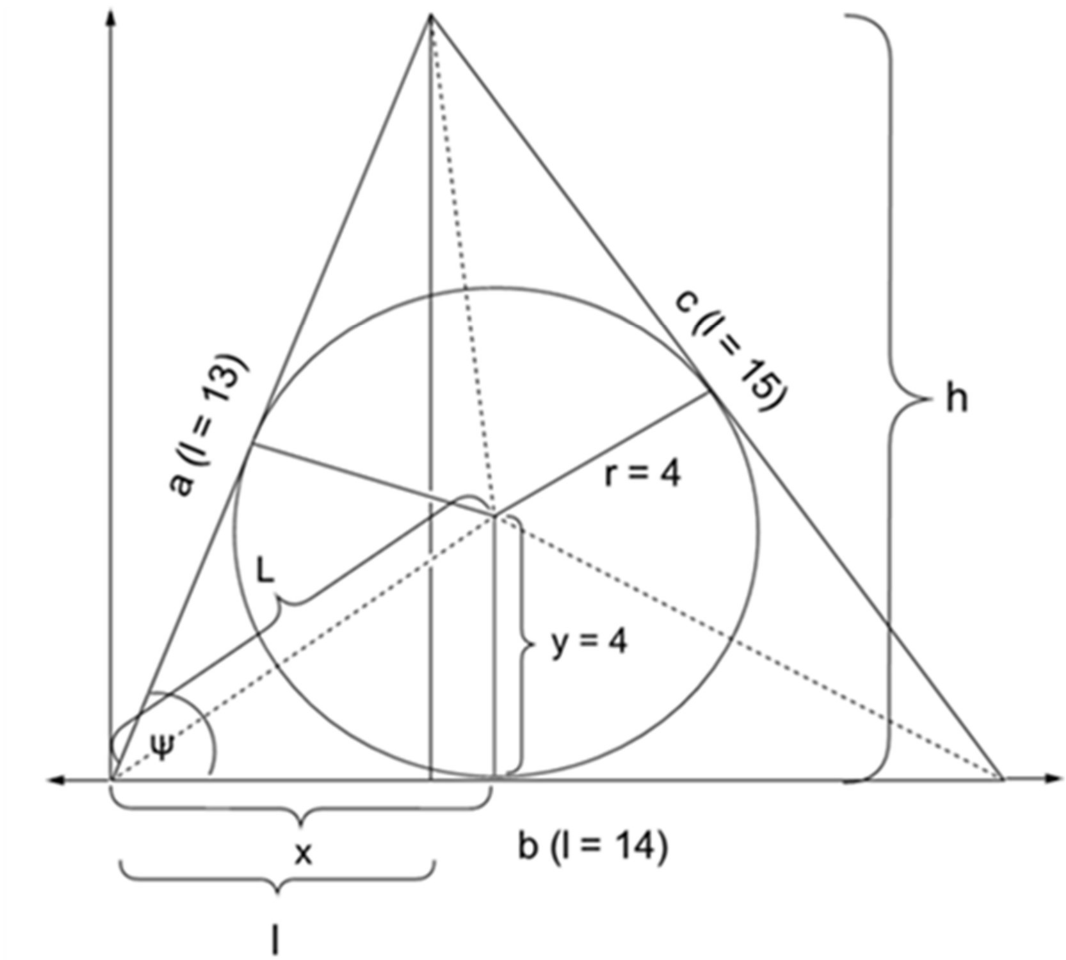
$$A = fr + gr + hr \qquad f + g + r = \frac{a+b+c}{2} = \frac{42}{2} = 21$$

$$A = 84 = r(f + g + h) = 21r \qquad r = 4$$

$$r = \frac{A}{s} \qquad s = \frac{a+b+c}{2}$$

3) The location of the incircle center:

To find the (x, y) position of the center of the incircle, consider the following:



$$\sin(\psi) = \frac{h}{a} = \frac{12}{13} \qquad \cos(\psi) = \frac{l}{a} = \frac{5}{13}$$

$$\sin\left(\frac{\psi}{2}\right) = \frac{r}{L} = \frac{4}{L} \qquad \cos\left(\frac{\psi}{2}\right) = \frac{x}{L}$$

$$\sin\left(\frac{\psi}{2}\right) = \sqrt{\frac{1-\cos(\psi)}{2}} = \sqrt{\frac{1-\frac{5}{13}}{2}} = 0.55470$$

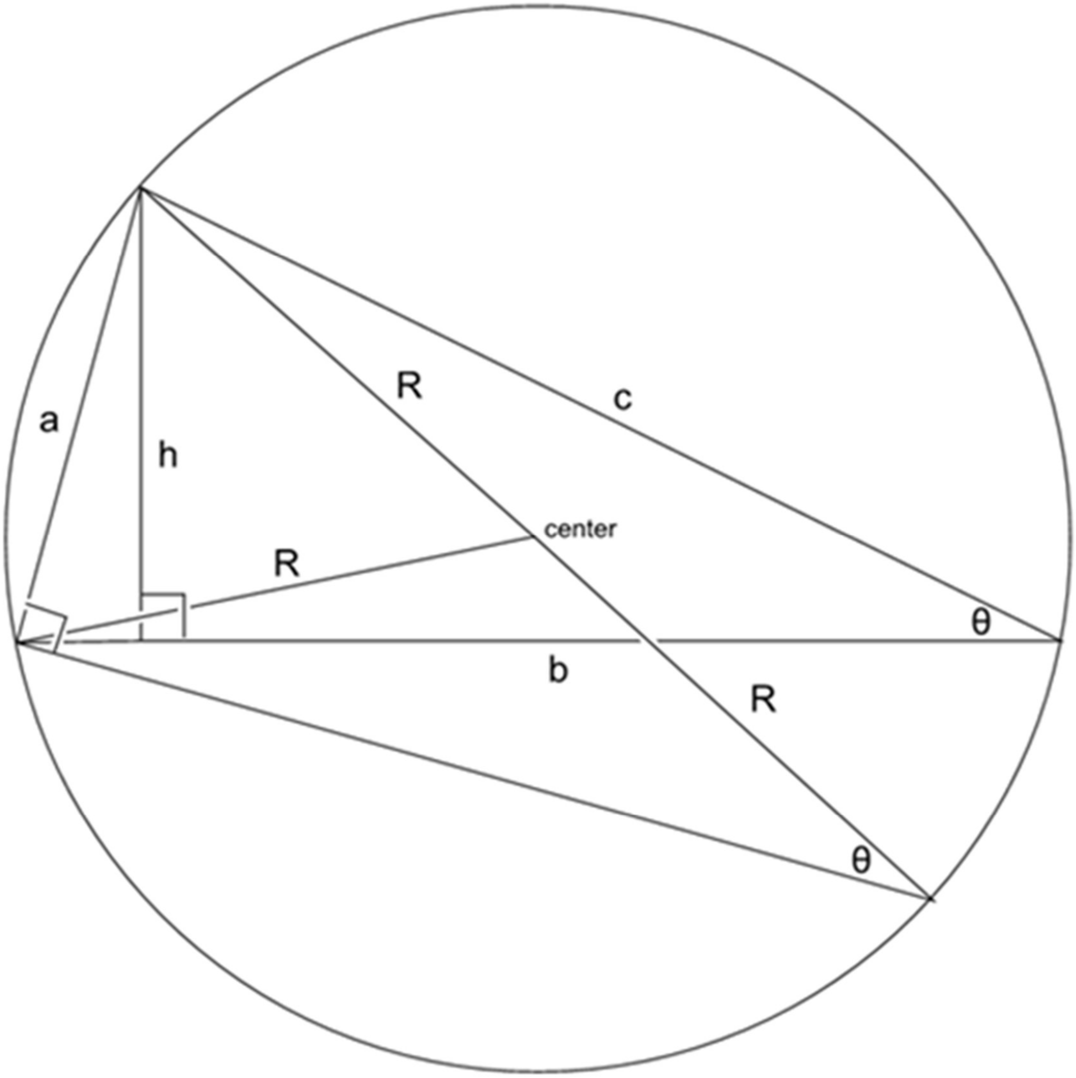
$$\cos\left(\frac{\psi}{2}\right) = \sqrt{\frac{1+\cos(\psi)}{2}} = \sqrt{\frac{1+\frac{5}{13}}{2}} = 0.83205$$

$$L = \frac{r}{\sin\left(\frac{\psi}{2}\right)} = \frac{4}{\sin\left(\frac{\psi}{2}\right)} = \frac{4}{0.55470} = 7.2111$$

$$x = L\cos\left(\frac{\psi}{2}\right) = (7.2111)(0.83205) = 6 \qquad y = r = 4$$

4) The radius of the circumcircle:

To calculate the radius of the triangle's circumcircle, consider the following diagram:



The Thales Central Angle Theorem guarantees a right angle in the triangle with a hypotenuse length of 2R (diameter of circle) given that R is the radius of the circle. The two triangles contain the same angle theta as guaranteed by the Thales Inscribed Angle Theorem; or equally logically by the fact that both angles sweep out the same arc within the circle.

$$\frac{c}{h} = \frac{2R}{a} \qquad Area = A = \frac{bh}{2} \qquad h = \frac{2A}{b}$$

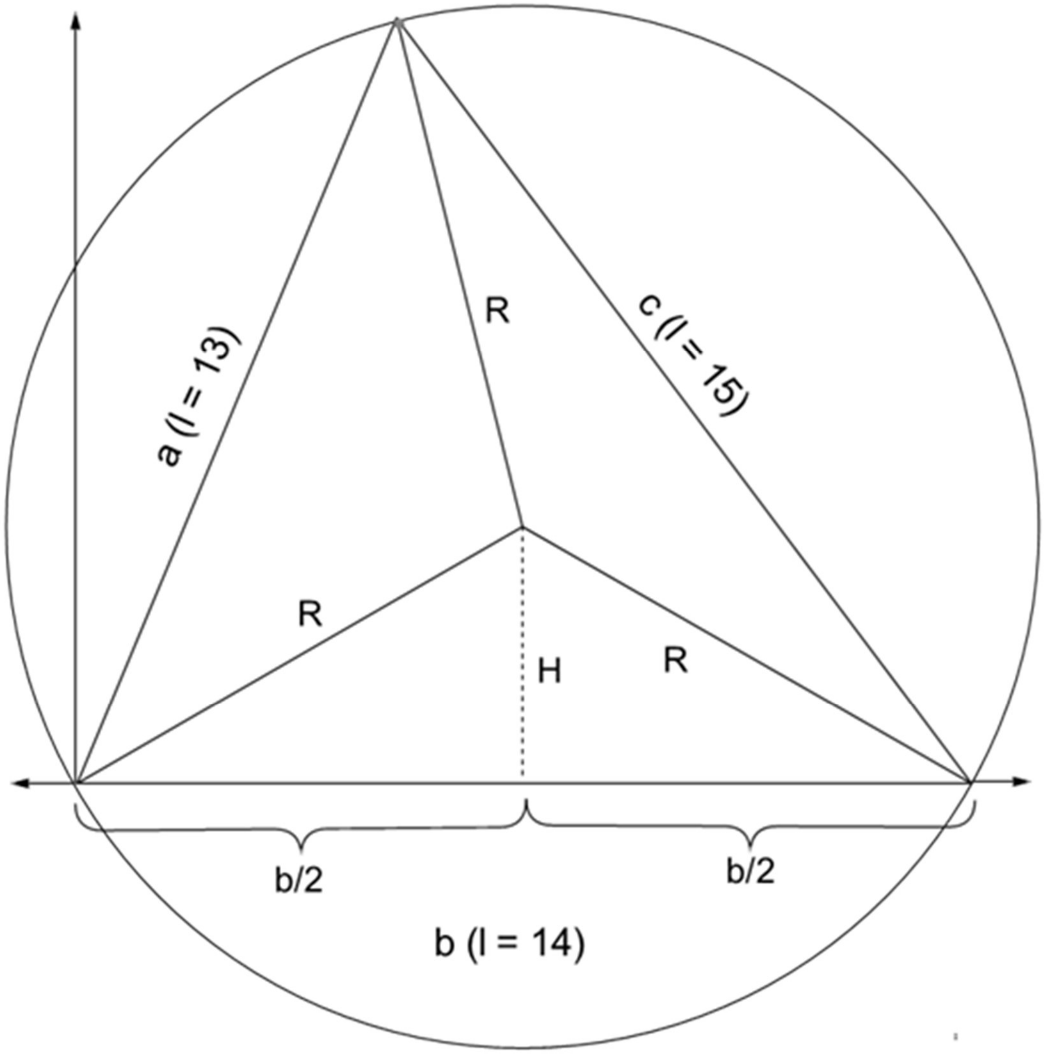
$$2R = \frac{ac}{\frac{2A}{b}} = \frac{abc}{2A} \qquad R = \frac{abc}{4A} = \frac{(13)(14)(15)}{(4)(84)} = 8.125$$

$$R = \frac{abc}{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}}$$

$$R = \frac{(13)(14)(15)}{\sqrt{(42)(16)(14)(12)}} = 8.125$$

5) The location of the circumcircle center:

To find the cartesian coordinates of the center of the circumcircle, consider the following diagram:



$$x = \frac{b}{2} = 7 \qquad y = H = \sqrt{R^2 - \frac{b^2}{4}} = \sqrt{(8.125)^2 - \frac{(14)^2}{4}} = 4.125$$

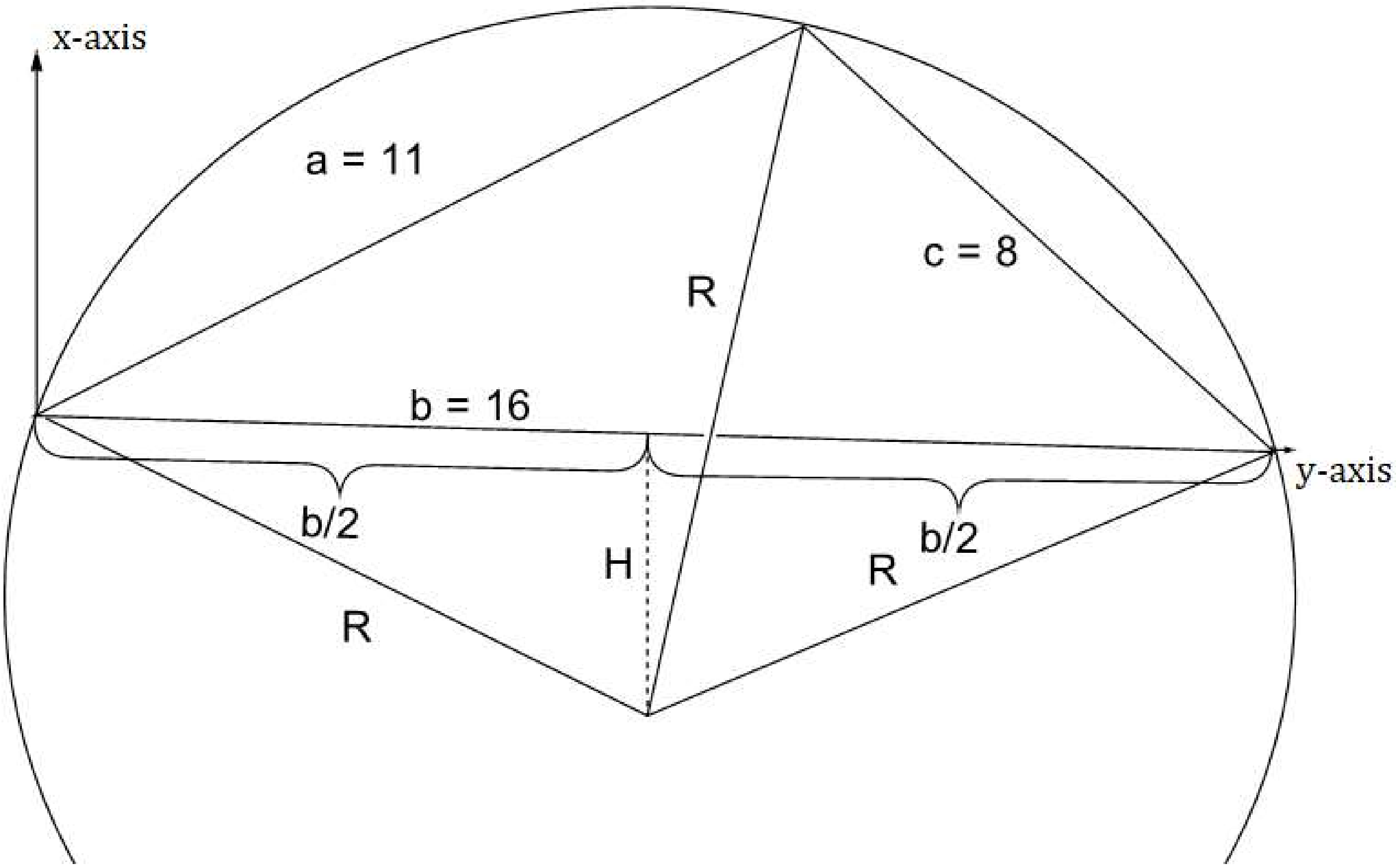
Note that the above equations for the (x, y) coordinates of the center is accurate so long as the center of the circumcircle is within the triangle. If the center of the circumcircle is outside the triangle, then, with the longest side of the triangle placed on the x-axis and the shortest side to the left (assign the longest side to side b and the shortest side to side a), the correct equations are given below. Note that this will not work if side "c" or side "a" are the longest (diagram above).

$$x = \frac{b}{2} \qquad y = -H = -\sqrt{R^2 - \frac{b^2}{4}} \qquad when\ \ b > \frac{a^2}{l}$$

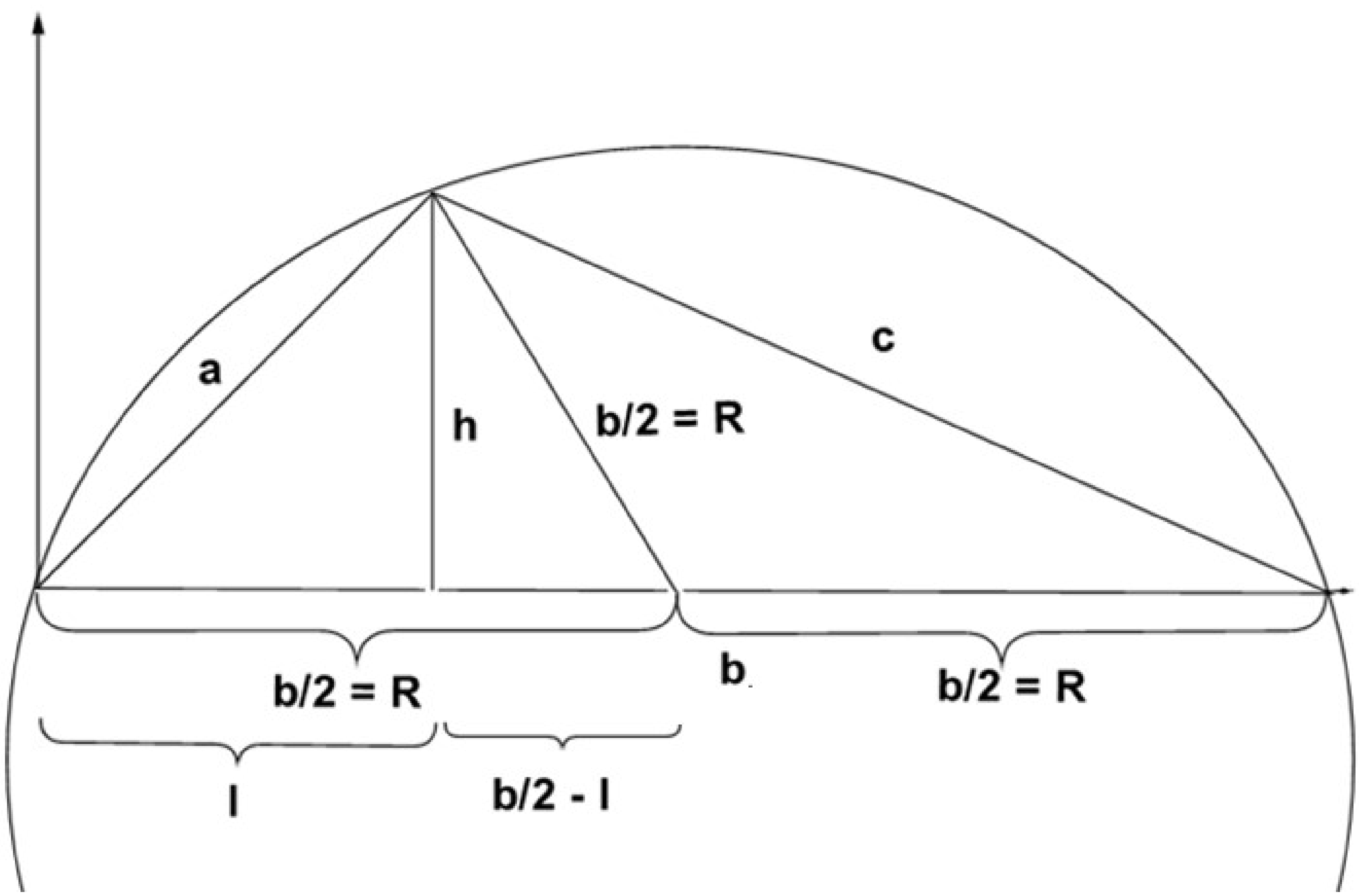
The center of the circumcircle is on the b side edge (*i.e.*, y coordinate of circumcircle center = 0) when:

$$\sqrt{\left(\frac{b}{2} - l\right)^2 + h^2} = \frac{b}{2} \qquad \left(\frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - bl + l^2 + h^2 \qquad b = \frac{a^2}{l}$$

The graphic below illustrates the case wherein the circumcircle center is outside the triangle.

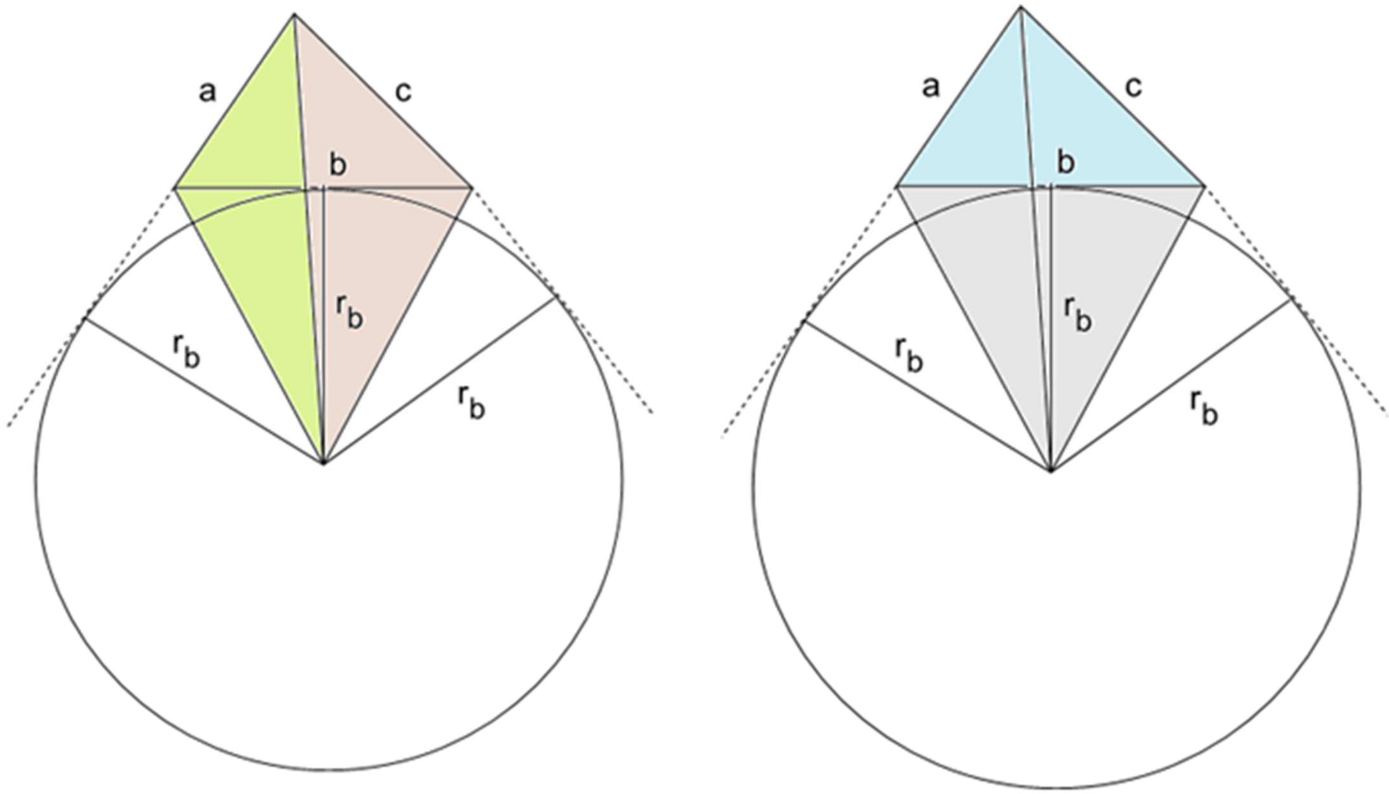


The graphic below illustrates the case wherein the circumcircle center is located on the b-side of the triangle.



6) The radius of the excircle:

To find the radius of an excircle of a triangle, consider the following diagram:



The total area of the quadrilateral in the lefthand figure is equal to the sum of the area of the two triangles (green & brown). The area of the original triangle (blue) is equal to the area of the quadrilateral minus the area of the grey triangle.

$$Area\ of\ Quadrilateral = \frac{ar_b + cr_b}{2}$$

$$Area\ of\ Triangle\ (blue) = A = \frac{ar_b + cr_b - br_b}{2} = \frac{r_b(a + c - b)}{2}$$

$$r_b = \frac{A}{\left(\frac{a + c - b}{2}\right)} = \frac{A}{\left(\frac{a + b + c - 2b}{2}\right)} = \frac{A}{\left(\frac{a + c - b}{2}\right)}$$

$$r_b = \frac{A}{s - a} \qquad s = \frac{a + b + c}{2}$$

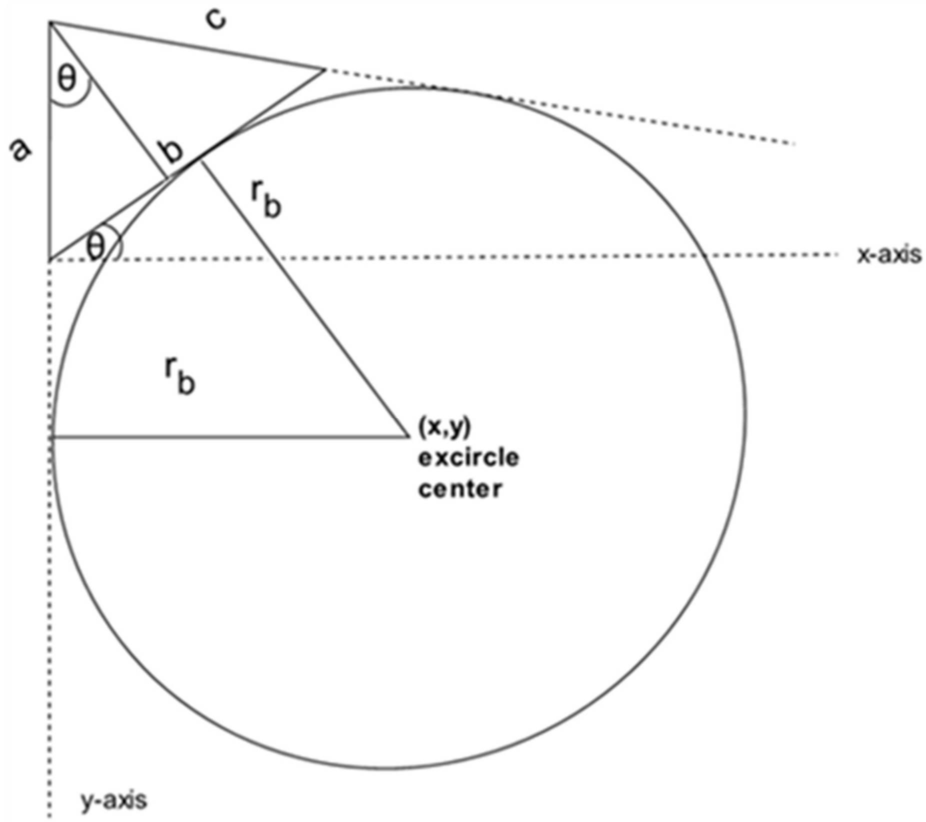
$$r_b = \frac{84}{(21 - 14)} = 12$$

The radii of the other two excircles are calculated in the same way.

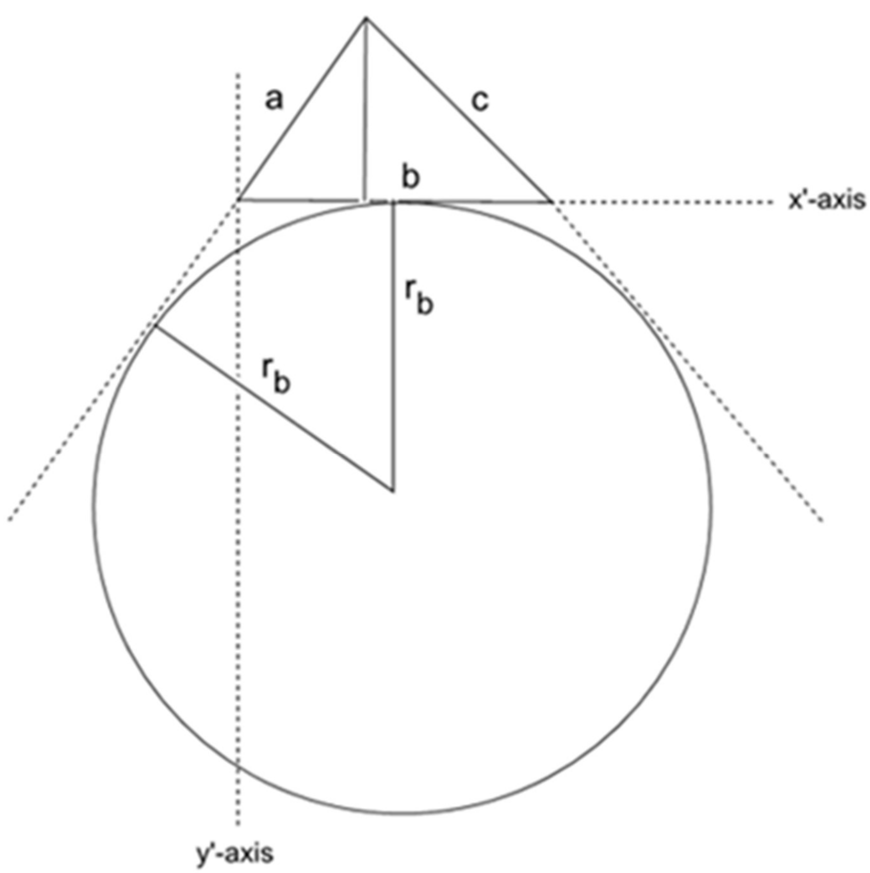
$$r_a = \frac{A}{s - a} = \frac{84}{(21 - 13)} = 10.5 \qquad r_c = \frac{A}{s - c} = \frac{84}{(21 - 15)} = 14$$

7) The location of the excircle center:

To find the cartesian coordinates of the center of the triangle's "b-side" excircle, consider the following diagram:



With the coordinate axes in this orientation, the value of x for the center of the excircle is r_b. We will leave the value of y undetermined for now. We next rotate the axes counterclockwise by the angle theta.



With the coordinate axes in this orientation, the value of y' is $-r_b$. We will calculate the value of x' as follows.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ -r_b \end{bmatrix} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} r_b \\ y \end{bmatrix}$$

$$x' = r_b\, cos(\theta) + ysin(\theta)$$

$$-r_b = -r_b\, sin(\theta) + ycos(\theta)$$

$$\frac{cos\,(\theta)}{sin\,(\theta)} x' = r_b \frac{cos^2(\theta)}{sin\,(\theta)} + ycos(\theta)$$

$$0 = r_b[1 - sin(\theta)] + ycos(\theta)$$

$$\frac{cos\,(\theta)}{sin\,(\theta)} x' = r_b \left\{ \frac{cos^2(\theta)}{sin\,(\theta)} + sin(\theta) - 1 \right\}$$

$$x' = r_b \left\{ \frac{cos^2(\theta)}{sin\,(\theta)} + sin(\theta) - 1 \right\} \frac{sin\,(\theta)}{cos\,(\theta)}$$

$$x' = r_b \left\{ \frac{1 - sin^2(\theta)}{sin(\theta)} + sin(\theta) - 1 \right\} tan(\theta)$$

$$x' = r_b \left\{ \frac{1}{sin(\theta)} - sin(\theta) + sin(\theta) - 1 \right\} tan(\theta)$$

$$x' = r_b\{csc(\theta) - 1\}tan(\theta)$$

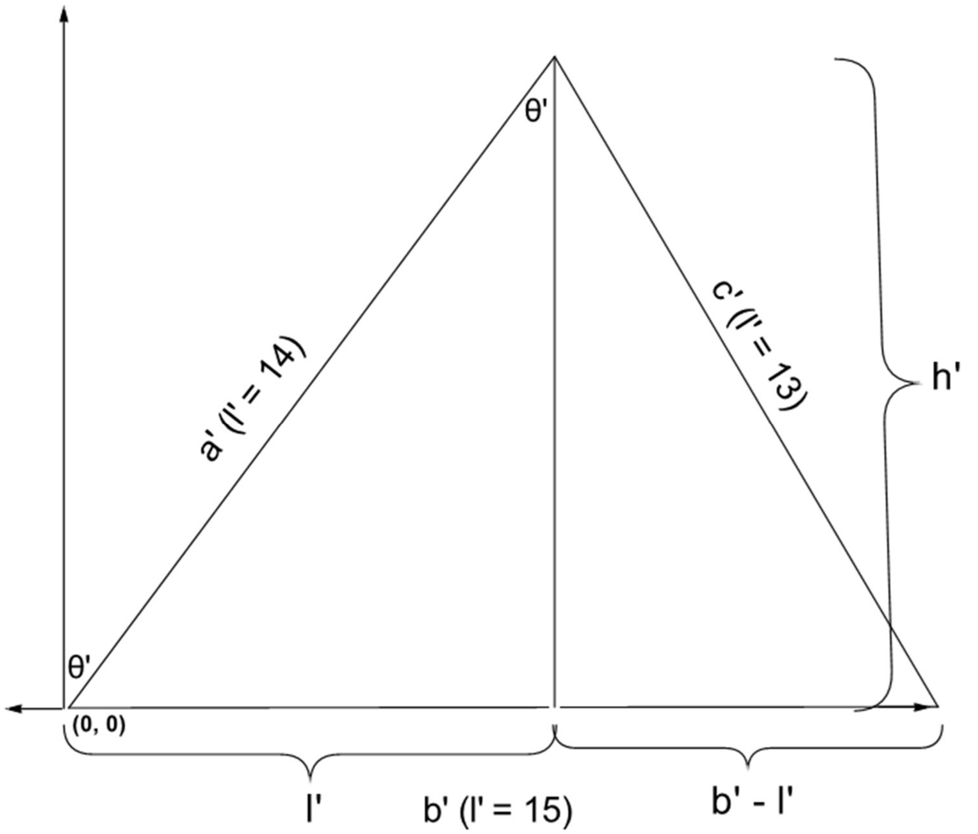
$$x' = r_b\{sec(\theta) - tan(\theta)\} = r_b \left(\frac{hypotenuse - opposite}{adjacent} \right)$$

$$x' = r_b \left\{ \frac{13}{12} - \frac{5}{12} \right\} = 12 \left(\frac{8}{12} \right) = 8$$

With the coordinate axes in the (x', y') orientation (*i.e.*, the orientation wherein the "b-side" of the triangle is oriented along the x'-axis), the coordinates of the center of the "b-side" excircle are (referring now to this orientation as x, y):

$$\boldsymbol{x = 8, \quad y = -12}$$

To find the center of the "c-side" excircle, we first exchange the triangle's sides as shown and find the center of the "b'-side" excircle:



$$l' = \frac{{a'}^2 + {b'}^2 - {c'}^2}{2b'} = 8.4 \qquad h' = \sqrt{a'^2 - \left(\frac{a'^2 + b'^2 - c'^2}{2b'}\right)^2} = 11.2$$

$$sec(\theta') = \frac{14}{11.2} = 1.25 \qquad tan(\theta') = \frac{8.4}{11.2} = 0.75 \qquad \theta' = 0.6435011$$

$$r_{b'} = r_c = \frac{A}{s-c} = \frac{A}{s-b'} = \frac{84}{(21-15)} = 14$$

$$x' = r_c\{\sec(\theta') - \tan(\theta')\} = r_c\{1.25 - 0.75\} = 14(0.5) = 7$$

$$x' = 7, \quad y' = -14$$

Now we want to rotate and translate this point (x', y') back into the original coordinate system (x, y). We need to rotate the point counterclockwise as a vector by an angle of $\pi/2 + \theta'$ followed by a translation of 14 in the positive x direction (translation = length of a').

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta' + \frac{\pi}{2}) & -\sin(\theta' + \frac{\pi}{2}) \\ \sin(\theta' + \frac{\pi}{2}) & \cos(\theta' + \frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 14 \\ 0 \end{bmatrix}$$

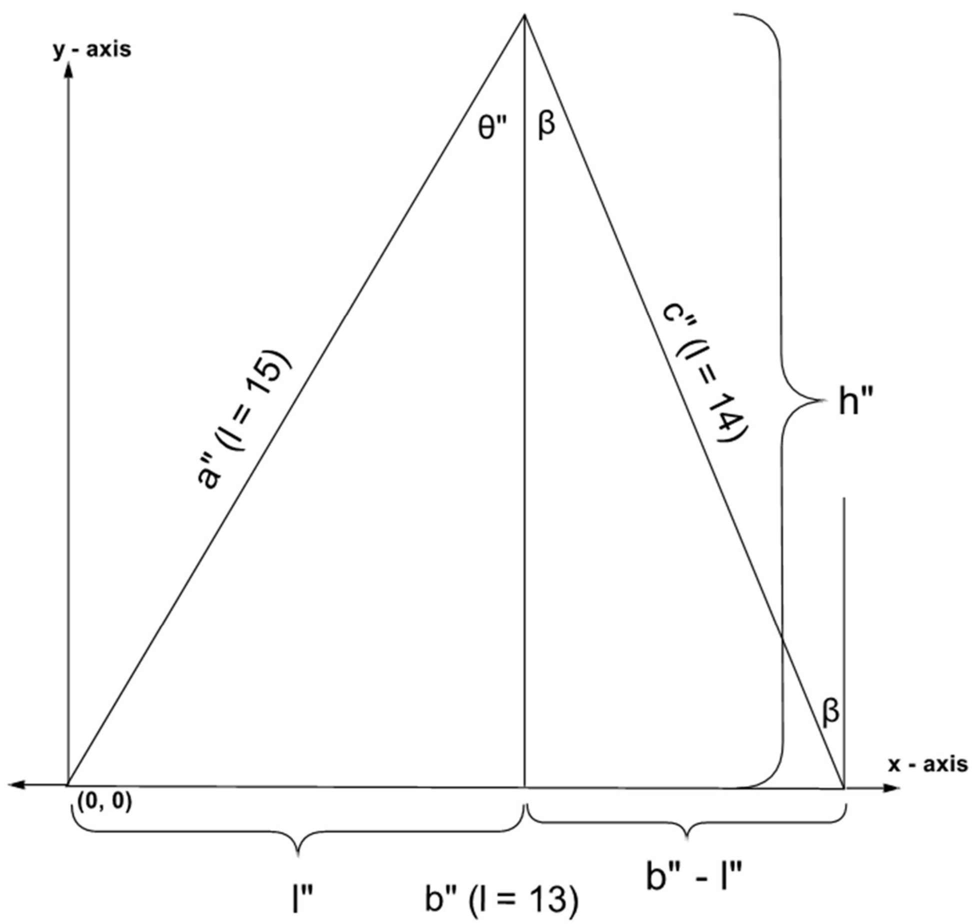
$$\begin{bmatrix}\cos(\theta' + \frac{\pi}{2}) & -\sin(\theta' + \frac{\pi}{2}) \\ \sin(\theta' + \frac{\pi}{2}) & \cos(\theta' + \frac{\pi}{2})\end{bmatrix} = \begin{bmatrix}\cos(2.214297) & -\sin(2.214297) \\ \sin(2.214297) & \cos(2.214297)\end{bmatrix}$$

$$\begin{bmatrix}\cos(2.214297) & -\sin(2.214297) \\ \sin(2.214297) & \cos(2.214297)\end{bmatrix}\begin{bmatrix}7 \\ -14\end{bmatrix} = \begin{bmatrix}-0.6 & -0.8 \\ 0.8 & -0.6\end{bmatrix}\begin{bmatrix}7 \\ -14\end{bmatrix}$$

$$\begin{bmatrix}x' \\ y'\end{bmatrix}_{rotated} = \begin{bmatrix}7 \\ 14\end{bmatrix}$$

$$\begin{bmatrix}x' \\ y'\end{bmatrix}_{rotated\,\&\,translated} = \begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}7+14 \\ 14\end{bmatrix} = \begin{bmatrix}21 \\ 14\end{bmatrix}$$

Finally, we reorient the triangle to find the center of the a-side (a-side = b"-side in reorientation) excircle.



$$l" = \frac{a"^2 + b"^2 - c"^2}{2b"} = 7.615385 \quad h" = \sqrt{a"^2 - \left(\frac{a"^2 + b"^2 - c"^2}{2b"}\right)^2} = 12.923077$$

$$sec(\theta") = \frac{a"}{h"} = \frac{15}{12.923077} = 1.160714$$

$$tan(\theta'') = \frac{l''}{h''} = \frac{7.615385}{12.923077} = 0.589286$$

$$\theta'' = 0.532504$$

$$r_{b''} = r_a = \frac{A}{s-a} = \frac{A}{s-b''} = \frac{84}{(21-13)} = 10.5$$

$$x'' = r_a\{sec(\theta'') - tan(\theta'')\} = 10.5(1.160714 - 0.589286) = 6$$

$$x'' = 6, \quad y'' = -10.5$$

In order to rotate this point (x", y") back into the original coordinate system (x, y), we also need the angle beta. We can find this angle with any inverse trigonometric function, and here we will use the inverse secant function along with a check using the inverse tangent function.

$$sec(\beta) = \frac{c''}{h''} = \frac{14}{12.923077} = 1.083333$$

$$tan(\beta) = \frac{b''-l''}{h''} = \frac{5.384615}{12.923077} = 0.416667$$

$$\beta = 0.3947914$$

Now we rotate and translate this point (x", y") back into the original coordinate system (x, y) by an initial translation of 13 (length of b") in the negative direction followed by a rotation of the translated point clockwise as a vector through an angle of $\pi/2 + \beta$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\beta + \frac{\pi}{2}) & \sin(\beta + \frac{\pi}{2}) \\ -\sin(\beta + \frac{\pi}{2}) & \cos(\beta + \frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} x'' - 13 \\ y'' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(1.965588) & \sin(1.965588) \\ -\sin(1.965588) & \cos(1.965588) \end{bmatrix} \begin{bmatrix} -7 \\ -10.5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.384615 & 0.92307671 \\ -0.92307671 & -0.384615 \end{bmatrix} \begin{bmatrix} -7 \\ -10.5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 10.5 \end{bmatrix}$$

8) Tangent points of the excircles with the triangle:

Note that we can find the tangent points of the excircles with the triangle as the intersection of a line from the center of the excircle that forms a right angle with the line containing the given side of the triangle.

For the intersection with the a-side; the line that contains the a-side of the triangle is:

$$y_a = m_a x + b_a = \frac{12-0}{5-0}x = \frac{12}{5}x = 2.4x$$

The line that contains the center of a-side excircle and forms a right angle with the a-side of the triangle:

$$y_{a,ex} = -\frac{1}{2.4}x + b_{a,ex} = -0.416667x + b_{a,ex}$$

$$b_{a,ex} = y_1 - mx_1 = 10.5 - (-0.416667)(-7) = 7.583331$$

$$y_{a,ex} = -0.416667x + 7.583331$$

Set things equal to find the intersection of the two lines:

$$2.4x = -0.416667x + 7.583331 \qquad x = \frac{7.583331}{2.816667} = 2.692307$$

$$y = -0.416667(2.692307) + 7.583331 = 6.461536$$

Tangent Point of the a-side Excircle with the a-side of the Triangle: (2.692307, 6.461536)

For the intersection with the b-side; the line that contains the b-side of the triangle is:

$$y_b = m_b x + b_b = 0$$

The line that contains the center of b-side excircle and also forms a right angle with the b-side of the triangle is a line parallel to the y-axis and containing the center of the b-side excircle; thus, the intersection occurs at (8, 0).

Tangent Point of the b-side Excircle with the b-side of the Triangle: (8, 0)

For the intersection with the c-side; the line that contains the c-side of the triangle is:

$$y_c = m_c x + b_c = \frac{h-0}{l-14}x + b_c = \frac{12-0}{5-14}x + b_c = -\frac{12}{9}x + b_c = -\frac{4}{3}x + b_c$$

$$b_c = y_n - mx_n = 0 - (-1.333333)(14) = 12 - (-1.333333)(5) = 18.666667$$

$$y_c = -1.333333x + 18.666667$$

The line that contains the center of c-side excircle and forms a right angle with the c-side of the triangle:

$$y_{c,ex} = -\frac{1}{-1.333333}x + b_{c,ex} = 0.75x + b_{c,ex}$$

$$b_{c,ex} = y_1 - mx_1 = 14 - 0.75(21) = -1.75$$

$$y_{c,ex} = 0.75x - 1.75$$

Set things equal to find the intersection of the two lines:

$$-1.333333x + 18.666667 = 0.75x - 1.75 \qquad x = 9.8$$

$$y = -1.333333(9.8) + 18.666667 = 0.75(9.8) - 1.75 \qquad y = 5.6$$

Tangent Point of the c-side Excircle with the c-side of the Triangle: (9.8, 5.6)

Note that the tangent point is not necessarily at the intersection of the line containing a given side of the triangle and the line from the center of the excircle on the same given side of the triangle and the center of the incircle.

We can also calculate these tangent points by rotation/translation of the tangent points of the excircles when they are oriented below the x-axis (*i.e.*, on the b-side of the original orientation) prior to rotation/translation. Here is the calculation for the tangent point of the c-side excircle. The center of the c-side excircle in the (x', y') coordinate system prior to rotation/translation is (7, -14), and the coordinates of this tangent point after rotation and translation back into the (x, y) orientation will be:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta' + \frac{\pi}{2}) & -\sin(\theta' + \frac{\pi}{2}) \\ \sin(\theta' + \frac{\pi}{2}) & \cos(\theta' + \frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} + \begin{bmatrix} 14 \\ 0 \end{bmatrix}$$

$$\theta' = 0.6435011$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(2.214297) & -\sin(2.214297) \\ \sin(2.214297) & \cos(2.214297) \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} + \begin{bmatrix} 14 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} + \begin{bmatrix} 14 \\ 0 \end{bmatrix} = \begin{bmatrix} 9.8 \\ 5.6 \end{bmatrix}$$

Here is the calculation for the tangent point of the a-side excircle. The center of the a-side excircle in the (x", y") coordinate system prior to rotation/translation is (6, 0), and the coordinates of this tangent point after rotation and translation back into the (x, y) orientation will be:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\beta + \frac{\pi}{2}) & \sin(\beta + \frac{\pi}{2}) \\ -\sin(\beta + \frac{\pi}{2}) & \cos(\beta + \frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} x'' - 13 \\ y'' \end{bmatrix}$$

$$\beta = 0.3947914$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(1.965588) & \sin(1.965588) \\ -\sin(1.965588) & \cos(1.965588) \end{bmatrix} \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.384615 & 0.92307671 \\ -0.92307671 & -0.384615 \end{bmatrix} \begin{bmatrix} -7 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.692305 \\ 6.461537 \end{bmatrix}$$

The tangent point with the b-side excircle in the original configuration does not require rotation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

One can also calculate the intersections of the lines from the centers of the excircles to the center of the circumcircle and the lines that contain the sides of the triangle (*i.e.*, line segments that are the sides of the triangle).

9) Intersections of triangle sides with line between circumcircle and excircle centers

Intersection of the a-side with the line between the center of the circumcircle and the center of the a-side excircle.

$$y_a = 2.4x$$

$$y_{a,circ} = \frac{(10.5 - 4.125)}{(-7 - 7)}x + b_{a,circ} = -0.4553571x + b_{a,circ}$$

$$y_n = -0.4553571x_n + b_{a,circ} \qquad b_{a,circ} = y_n + 0.4553571x_n$$

$$b_{a,circ} = 10.5 + (0.4553571)(-7) = 4.125 + (0.4553571)(7) = 7.3125$$

$$y_{a,circ} = -0.4553571x + 7.3125$$

The intersection of the two lines is:

$$2.4x = -0.4553571x + 7.3125 \qquad x_{intersect} = 2.560976$$

$$y_{intersect} = -0.4553571x_{intersect} + 7.3125 = 2.4x_{intersect} = 6.146341$$

Intersection of line between <u>a-side Excircle center and Circumcircle center</u> & a-side = (2.560976, 6.146341)

Intersection of the b-side with the line between the center of the circumcircle and the center of the b-side excircle.

$$y_b = 0$$

$$y_{b,circ} = \frac{(-12 - 4.125)}{(8 - 7)}x + b_{b,circ} = -16.125x + b_{b,circ}$$

$$b_{b,circ} = -12 + (16.125)(8) = 4.125 + (16.125)(7) = 117$$

$$y_{b,circ} = -16.125x + 117$$

The value of y along the b-side of the triangle is zero.

$$0 = -16.125x + 117 \qquad x = 7.255814$$

Intersection of line between <u>b-side Excircle center and Circumcircle center</u> & b-side = (7.255814, 0)

Intersection of the c-side with the line between the center of the circumcircle and the center of the c-side excircle.

$$y_c = -1.333333x + 18.666667$$

$$y_{c,circ} = \frac{(14 - 4.125)}{21 - 7)}x + b_{c,circ} = 0.705357x + b_{c,circ}$$

$$b_{c,circ} = 14 - (0.705357)(21) = 4.125 - (0.705357)(7) = -0.812498$$

$$y_{c,circ} = 0.705357x - 0.812498$$

The intersection of the two lines is:

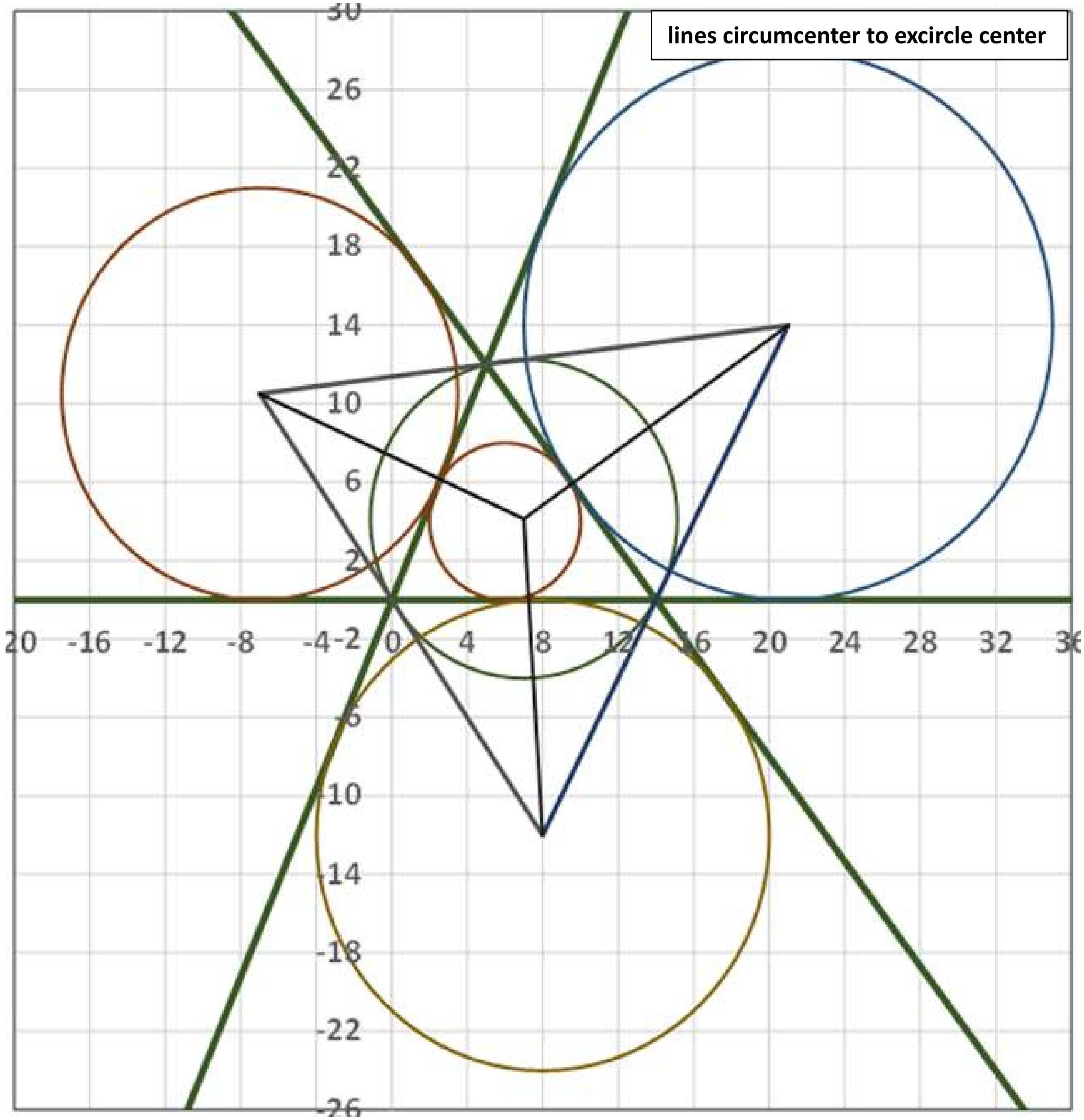
$$-1.333333x + 18.666667 = 0.705357x - 0.812498 \qquad x_{intersect} = 9.554746$$

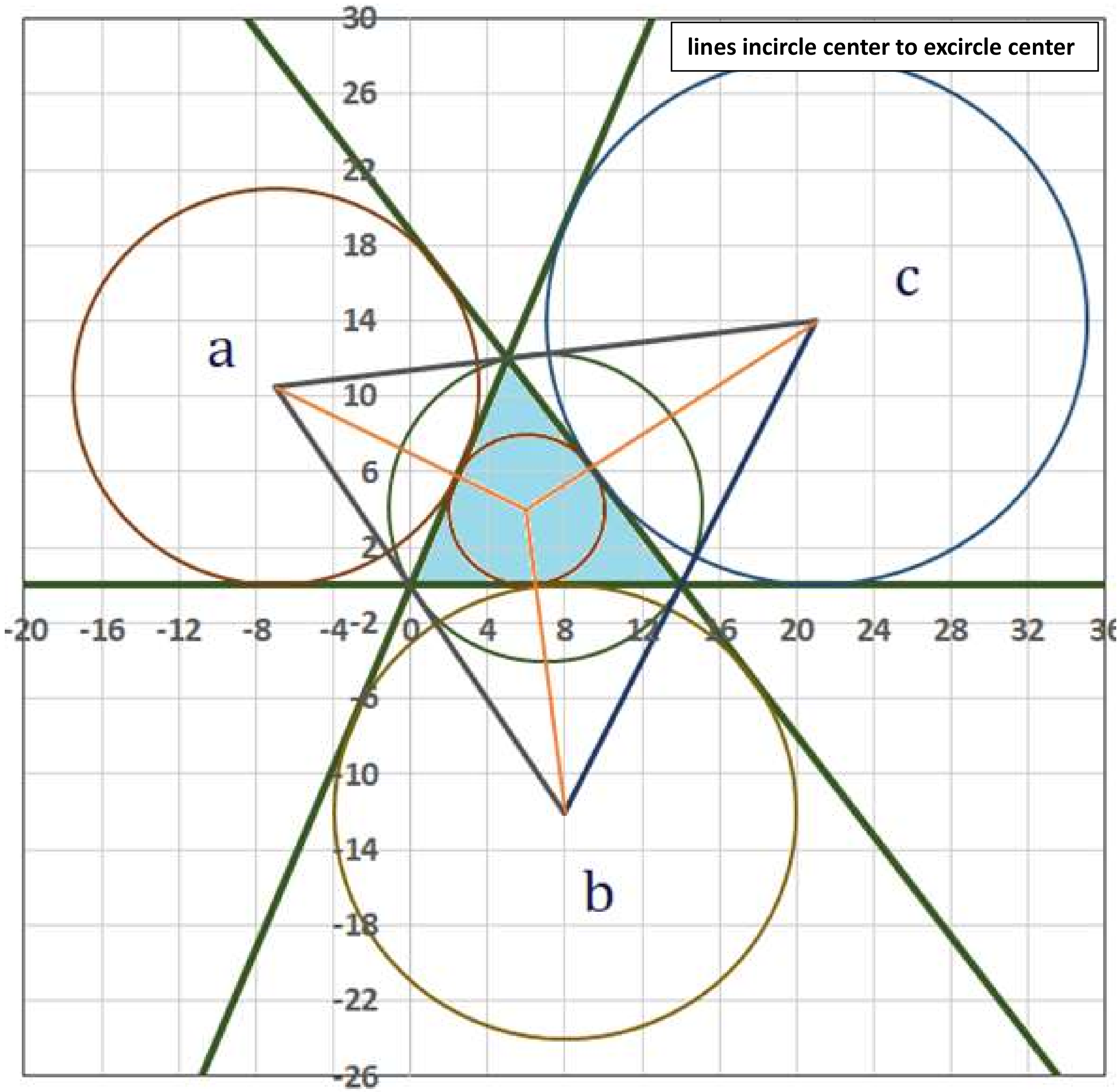
$$y_{intersect} = -1.333333x_{intersect} + 18.666667 = 5.927009$$

$$y_{intersect} = 0.705357x_{intersect} - 0.812498 = 5.927009$$

Intersection of line between b-side Excircle center and Circumcircle center & b-side = (9.554746, 5.927009)

Compare the two graphics below. The line between the circumcircle center and a given excircle center intersect the adjacent side of the triangle closer to the tangent point of the given excircle with the adjacent side than does the corresponding line between the incircle center and the given excircle center; but neither set of intersections align exactly with the tangent points of the excircles in a scalene triangle.





10) Distances between excircle centers:

The distances (d) between the excircle centers of a 13, 14, 15 triangle are:

$$d_{a/b} = \sqrt{(-7-8)^2 + (10.5-(-12))^2} = \sqrt{(-15)^2 + (22.5)^2} = 27.0416$$

$$d_{a/c} = \sqrt{(-7-21)^2 + (10.5-14)^2} = \sqrt{(-28)^2 + (-3.5)^2} = 28.2179$$

$$d_{b/c} = \sqrt{8-21)^2 + (-12-14)^2} = \sqrt{(-13)^2 + (-28)^2} = 30.8707$$

The distances between the incircle center and the excircle centers of a 13, 14, 15 triangle are:

$$d_{a/in} = \sqrt{(6-(-7))^2 + (4-10.5)^2} = \sqrt{(13)^2 + (-6.5)^2} = 14.5344$$

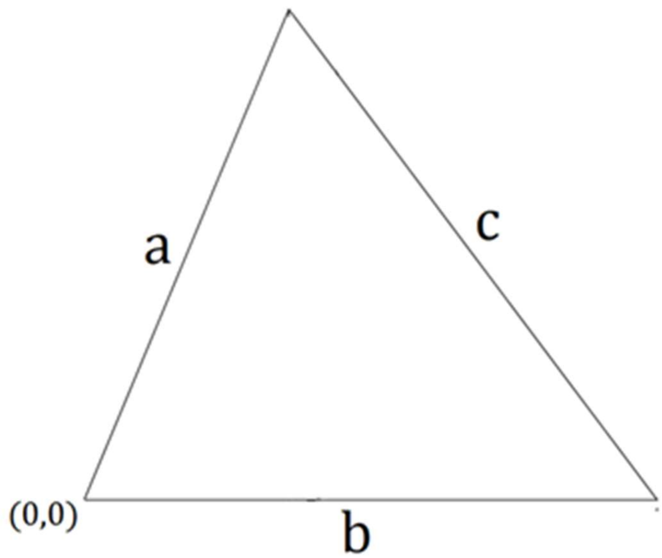
$$d_{b/in} = \sqrt{6-8)^2 + (4-(-12))^2} = \sqrt{(-2)^2 + (16)^2} = 16.1245$$

$$d_{c/in} = \sqrt{6-21)^2 + (4-14)^2} = \sqrt{(-15)^2 + (-10)^2} = 18.0278$$

The distance between the incircle center and the circumcircle center of a 13, 14, 15 triangle is:

$$d_{in/cir} = \sqrt{6-7)^2 + (4-4.125)^2} = \sqrt{(-1)^2 + (-0.125)^2} = 1.007782$$

11) Summary of 13, 14, 15 triangle parameters with the (13, 14) vertex at the origin (a = 13, b = 14, c = 15):

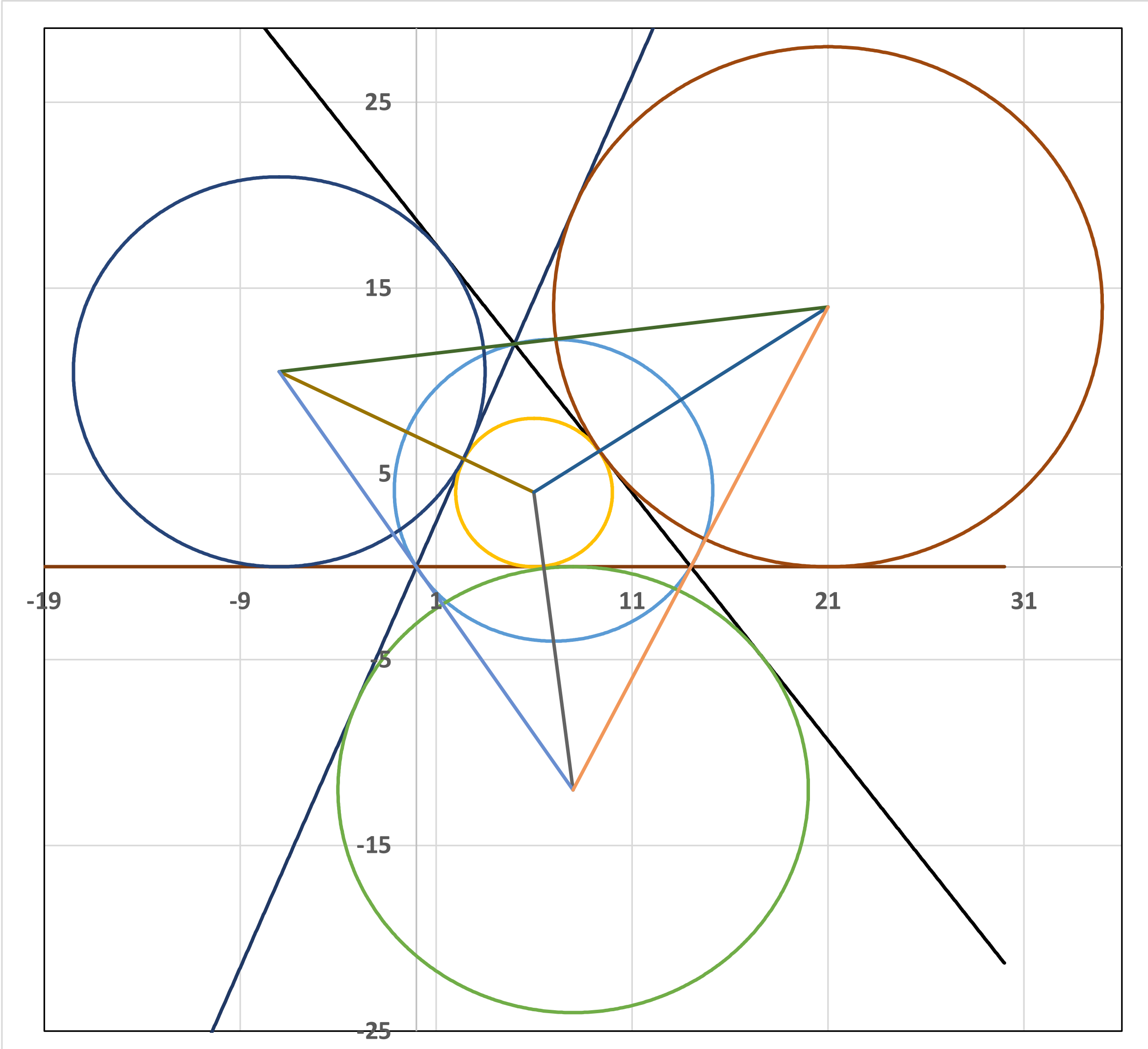


Area = 84
Coordinates of (a, b) vertex = (0, 0)
Coordinates of (b, c) vertex = (14, 0)
Coordinates of (a, c) vertex = (5, 12)
Radius of Incircle = 4
Center of Incircle = (6, 4)
Radius of Circumcircle = 8.125
Center of Circumcircle = (7, 4.125)
Radius of a-side (13) Excircle = 10.5
Center of a-side Excircle = (-7, 10.5)
Tangent Point of a-side excircle = (2.692307, 6.461536)
Radius of b-side (14) Excircle = 12
Center of b-side Excircle = (8, -12)
Tangent Point of b-side excircle = (8, 0)
Radius of c-side Excircle = 14
Center of c-side Excircle = (21, 14)
Tangent Point of c-side excircle = (9.8, 5.6)

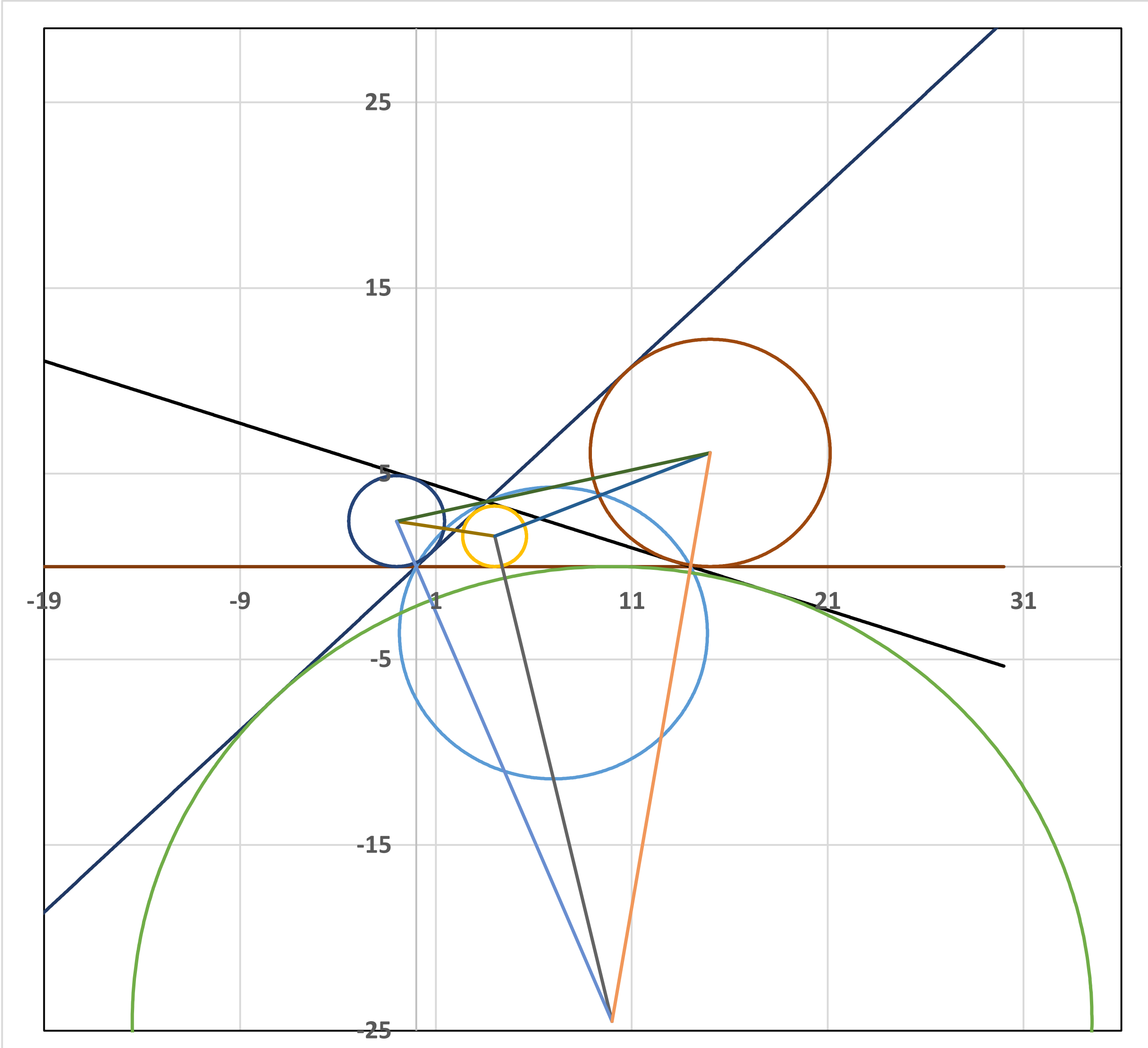
If you are looking for a good "non-right" triangle to illustrate numerous calculations, then you can't do better than the 13, 14, 15 triangle. The 13, 14, 15 scalene triangle is the functional analog of the 3, 4, 5 right triangle in terms of ease of use. Observe that the lines between two given excircle centers pass through the corresponding vertex of the triangle. In a scalene triangle, the line from the incircle center or the circumcircle center to a given excircle center do not pass through the tangent point of said given excircle with the triangle.

12) Graphics and parameter summaries for a variety of triangles:

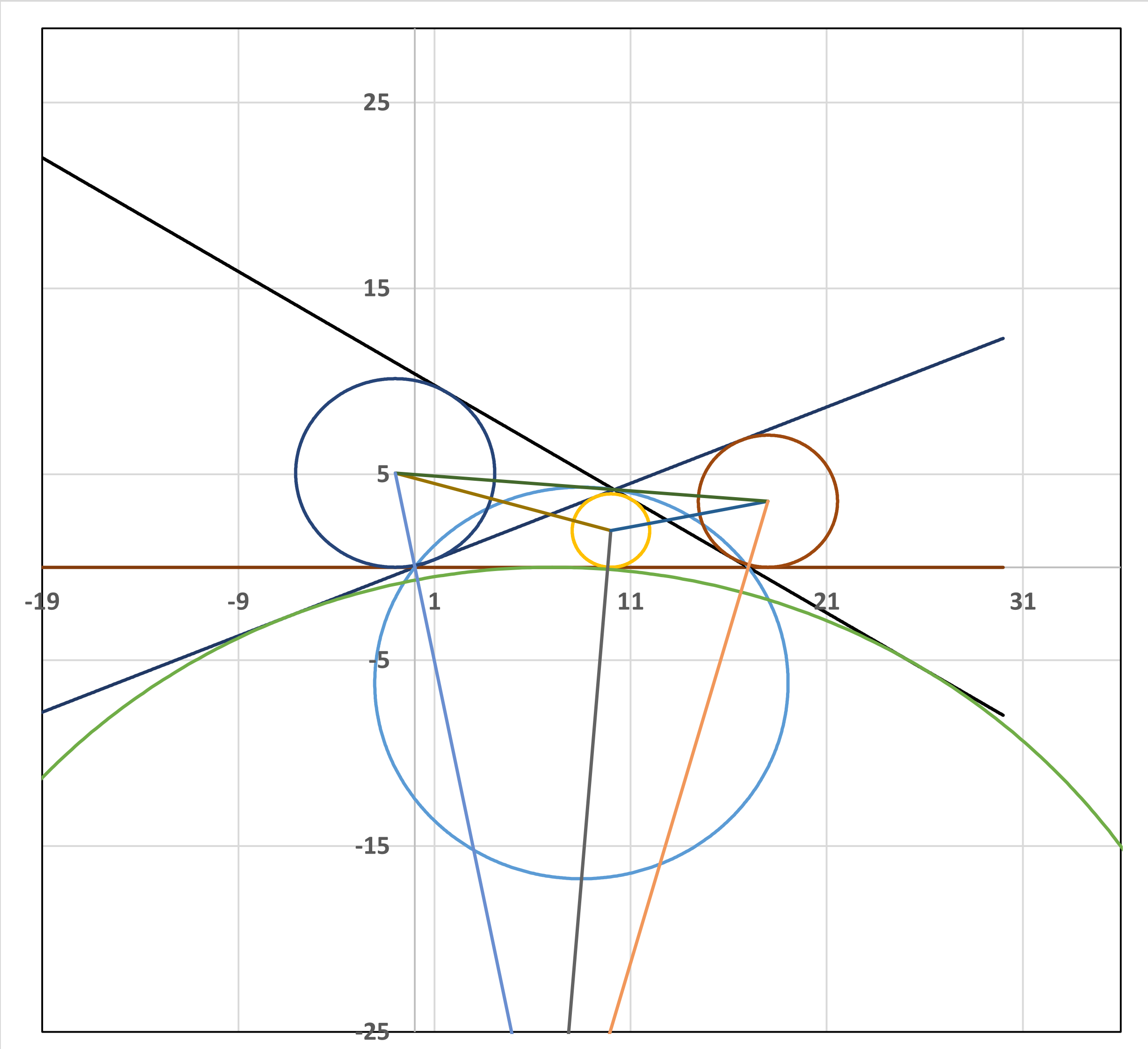
An Excel sheet was composed to produce triangle diagrams (lines to incircle center) for arbitrary isosceles, equilateral, and scalene triangles. Below are diagrams and parameters summaries for several different triangles.



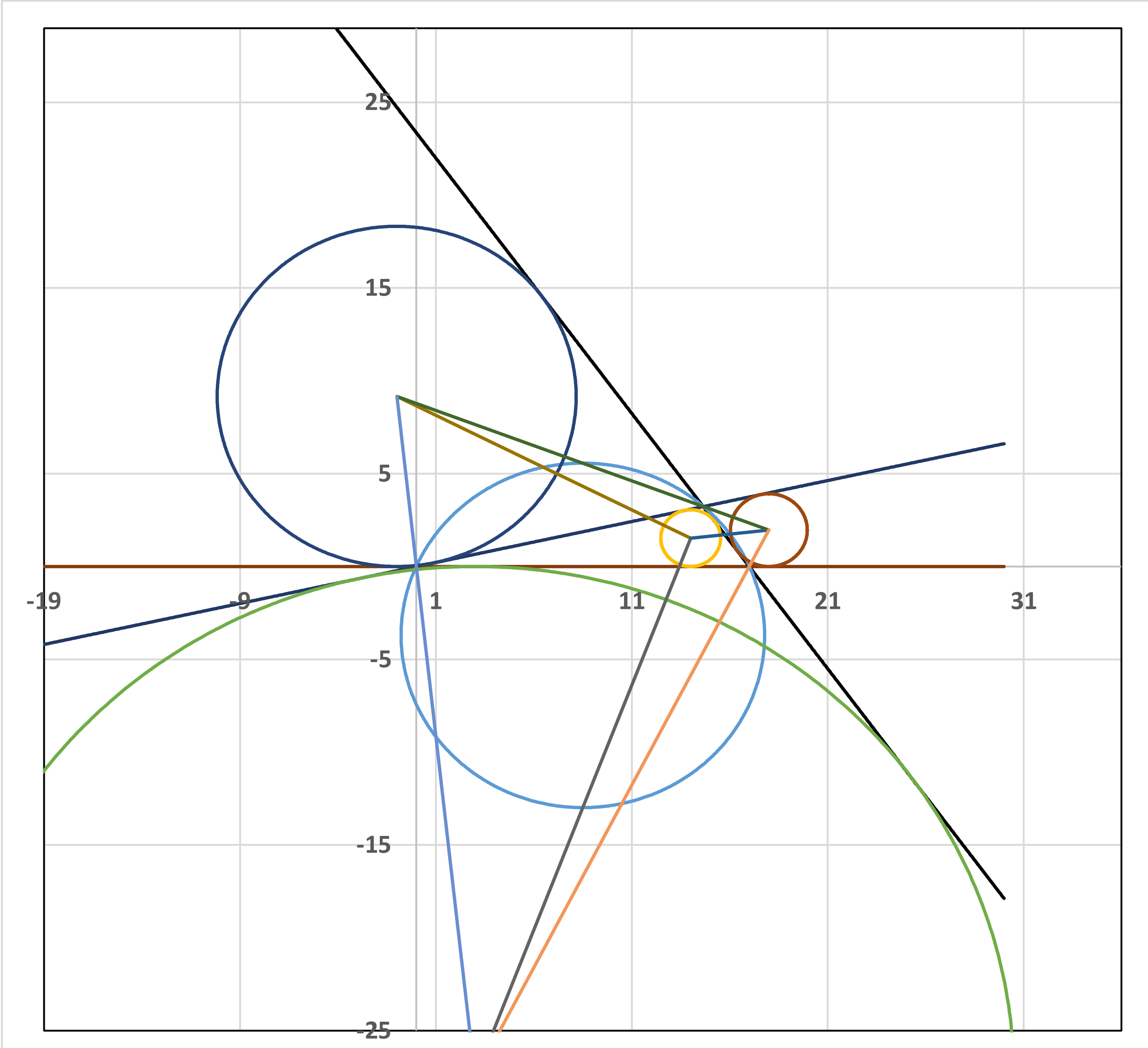
a	13
b	14
c	15
Triangle Area	84
length (a down)	7.615384615
length (b down)	5
length (c down)	8.4
height (a down)	12.92307692
height (b down)	12
height (c down)	11.2
q (a down) - q" (radians)	0.532504098
q (b down) (radians)	0.39479112
q (c down) - q' (radians)	0.643501109
Sum of Angles (degrees)	90
Incircle Radius	4
Circumcircle Radius	8.125
a side Excircle Radius	10.5
b side Excircle Radius	12
c side Excircle Radius	14
Center (Incircle)	6, 4
Center (Circumcircle)	7, 4.125
Center (Excircle a)	-7, 10.5
Center (Excircle b)	8, -12
Center (Excircle c)	21, 14



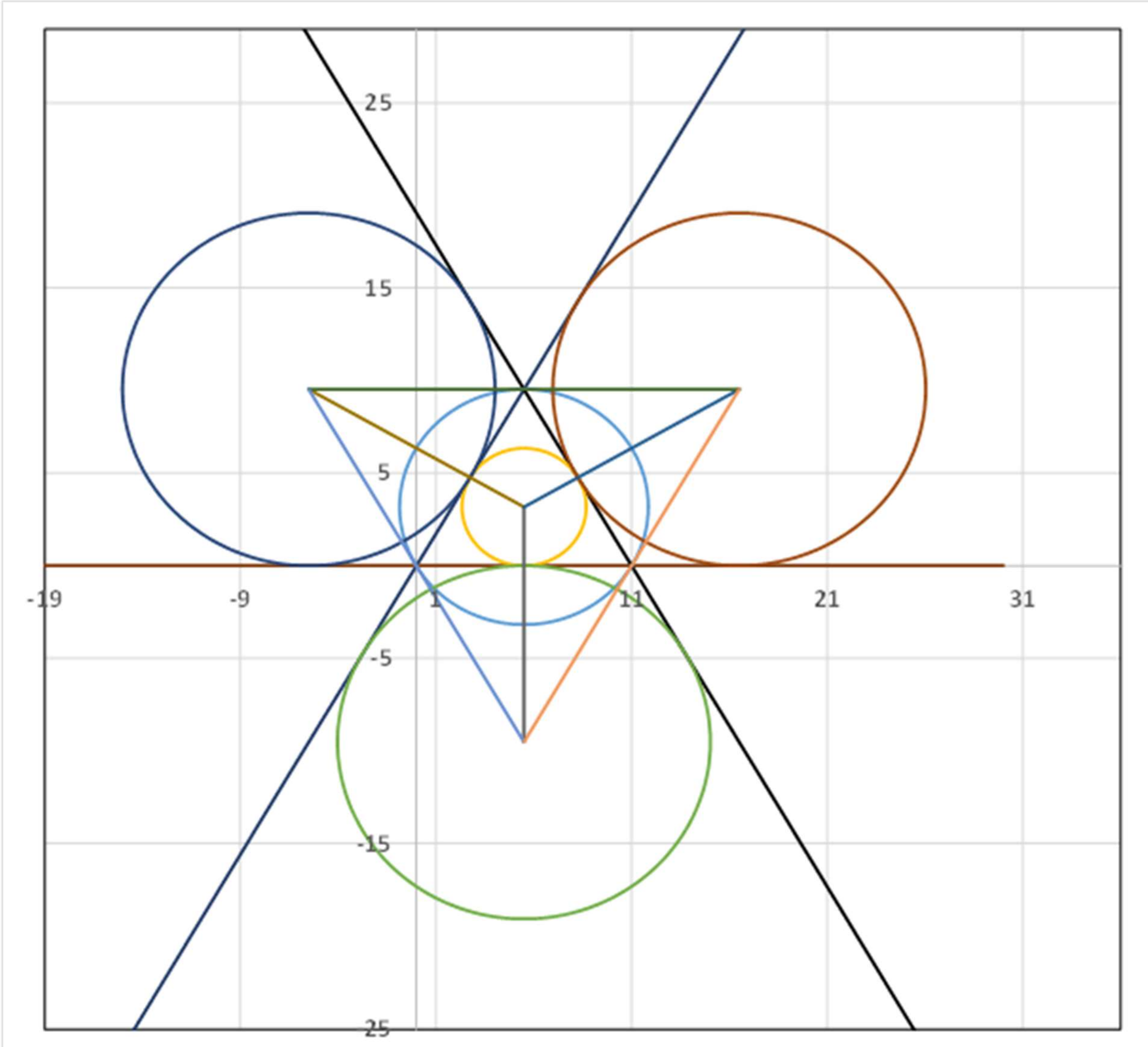
a	5
b	14
c	11
Triangle Area	24.49489743
length (a down)	-5
length (b down)	3.571428571
length (c down)	13.27272727
height (a down)	9.797958971
height (b down)	3.499271061
height (c down)	4.453617714
q (a down) - q" (radians)	-0.471861837
q (b down) (radians)	0.795602953
q (c down) - q' (radians)	1.247055211
Sum of Angles (degrees)	90
Incircle Radius	1.632993162
Circumcircle Radius	7.858779591
a side Excircle Radius	2.449489743
b side Excircle Radius	24.49489743
c side Excircle Radius	6.123724357
Center (Incircle)	4, 1.632993162
Center (Circumcircle)	7, -3.572172542
Center (Excircle a)	-1, 2.449489743
Center (Excircle b)	10, -24.49489743
Center (Excircle c)	15, 6.123724357



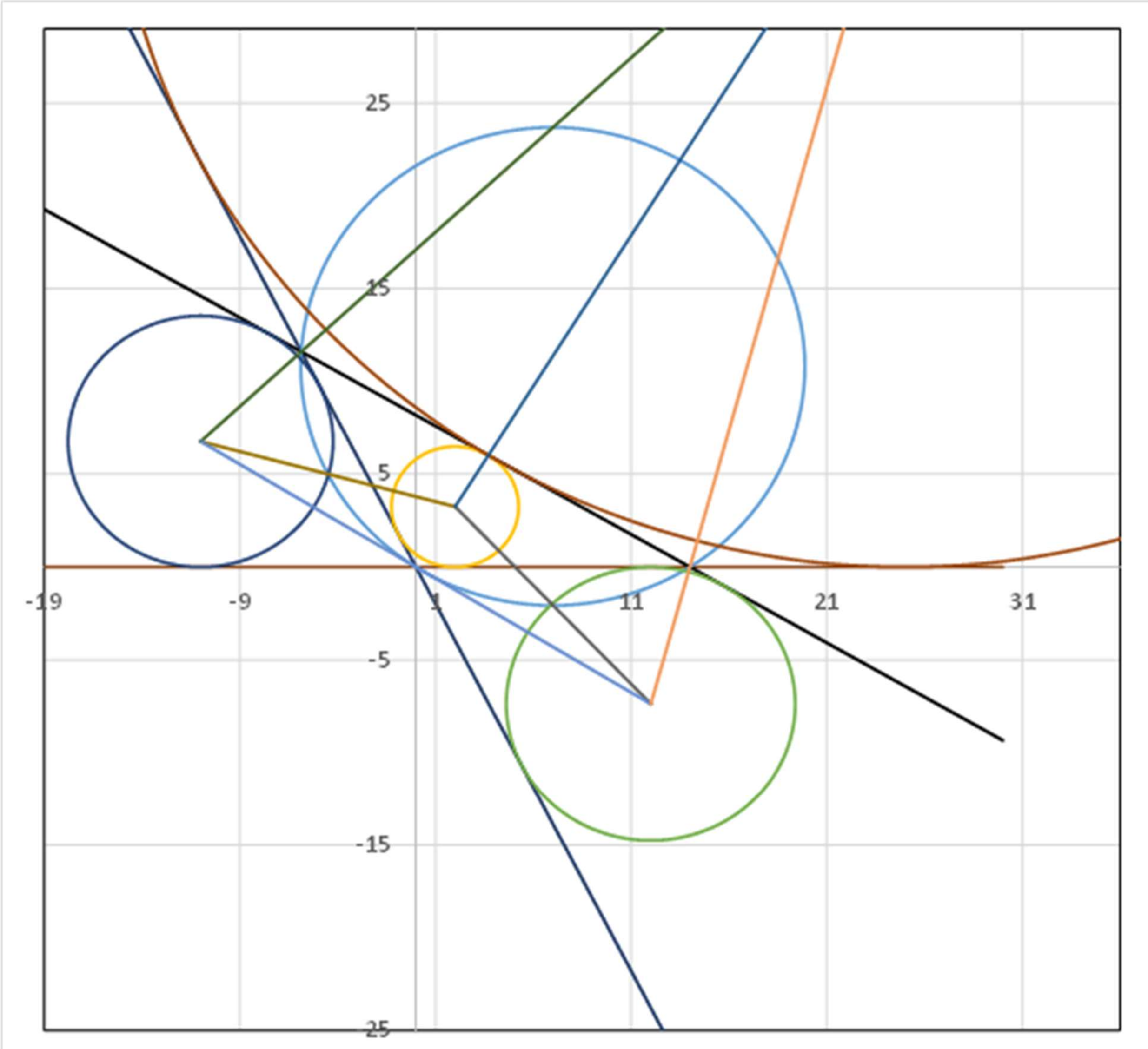
a	11
b	17
c	8
Triangle Area	35.4964787
length (a down)	-4.727272727
length (b down)	10.17647059
length (c down)	14.5
height (a down)	6.453905218
height (b down)	4.176056317
height (c down)	8.874119675
q (a down) - q" (radians)	-0.632185249
q (b down) (radians)	1.181387591
q (c down) - q' (radians)	1.021593985
Sum of Angles (degrees)	90
Incircle Radius	1.972026594
Circumcircle Radius	10.53625638
a side Excircle Radius	5.070925528
b side Excircle Radius	35.4964787
c side Excircle Radius	3.54964787
Center (Incircle)	10, 1.972026594
Center (Circumcircle)	8.5, -6.225969677
Center (Excircle a)	-1, 5.070925528
Center (Excircle b)	7, -35.4964787
Center (Excircle c)	18, 3.54964787



a	15
b	17
c	4
Triangle Area	27.49545417
length (a down)	-1.6
length (b down)	14.64705882
length (c down)	10
height (a down)	3.666060556
height (b down)	3.234759314
height (c down)	13.74772708
q (a down) - q" (radians)	-0.411516846
q (b down) (radians)	1.353438247
q (c down) - q' (radians)	0.628874925
Sum of Angles (degrees)	90
Incircle Radius	1.527525232
Circumcircle Radius	9.274260335
a side Excircle Radius	9.16515139
b side Excircle Radius	27.49545417
c side Excircle Radius	1.963961012
Center (Incircle)	14, 1.527525232
Center (Circumcircle)	8.5, -3.709704134
Center (Excircle a)	-1, 9.16515139
Center (Excircle b)	3, -27.49545417
Center (Excircle c)	18, 1.963961012



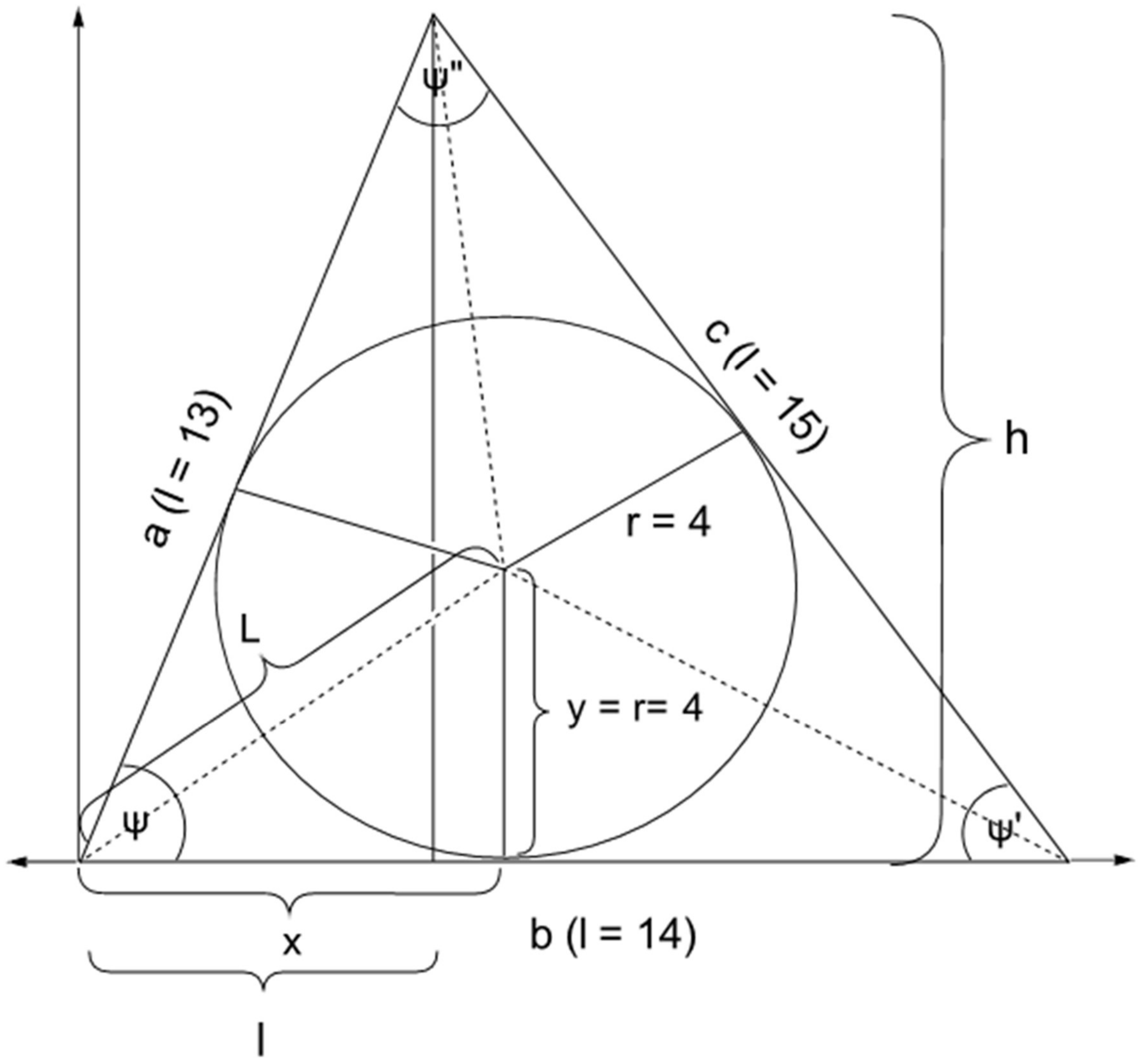
a	11
b	11
c	11
Triangle Area	52.39453693
length (a down)	5.5
length (b down)	5.5
length (c down)	5.5
height (a down)	9.526279442
height (b down)	9.526279442
height (c down)	9.526279442
q (a down) - q" (radians)	0.523598776
q (b down) (radians)	0.523598776
q (c down) - q' (radians)	0.523598776
Sum of Angles (degrees)	90
Incircle Radius	3.175426481
Circumcircle Radius	6.350852961
a side Excircle Radius	9.526279442
b side Excircle Radius	9.526279442
c side Excircle Radius	9.526279442
Center (Incircle)	5.5, 3.175426481
Center (Circumcircle)	5.5, 3.175426481
Center (Excircle a)	-5.5, 9.526279442
Center (Excircle b)	5.5, -9.526279442
Center (Excircle c)	16.5, 9.526279442



a	13
b	14
c	23
Triangle Area	81.24038405
length (a down)	19.30769231
length (b down)	-5.857142857
length (c down)	12.08695652
height (a down)	12.49852062
height (b down)	11.60576915
height (c down)	7.064381221
q (a down) - q" (radians)	0.99629774
q (b down) (radians)	-0.467380701
q (c down) - q' (radians)	1.041879288
Sum of Angles (degrees)	90
Incircle Radius	3.249615362
Circumcircle Radius	12.88152453
a side Excircle Radius	6.770032004
b side Excircle Radius	7.385489459
c side Excircle Radius	40.62019202
Center (Incircle)	2, 3.249615362
Center (Circumcircle)	7, 10.81358748
Center (Excircle a)	-11, 6.770032004
Center (Excircle b)	12, -7.385489459
Center (Excircle c)	25, 40.62019202

Appendix I: Tangent Points of the Incircle with the Triangle

The tangent points of the incircle with the triangle can be determined in a manner similar to that used to find the (x, y) coordinates of the excircles. The tangent point of the incircle with the b-side of the triangle is the easiest.



$$\sin(\psi) = \frac{h}{a} = \frac{12}{13} \qquad \cos(\psi) = \frac{l}{a} = \frac{5}{13}$$

$$\sin\left(\frac{\psi}{2}\right) = \frac{r}{L} = \frac{4}{L} \qquad cos\left(\frac{\psi}{2}\right) = \frac{x}{L}$$

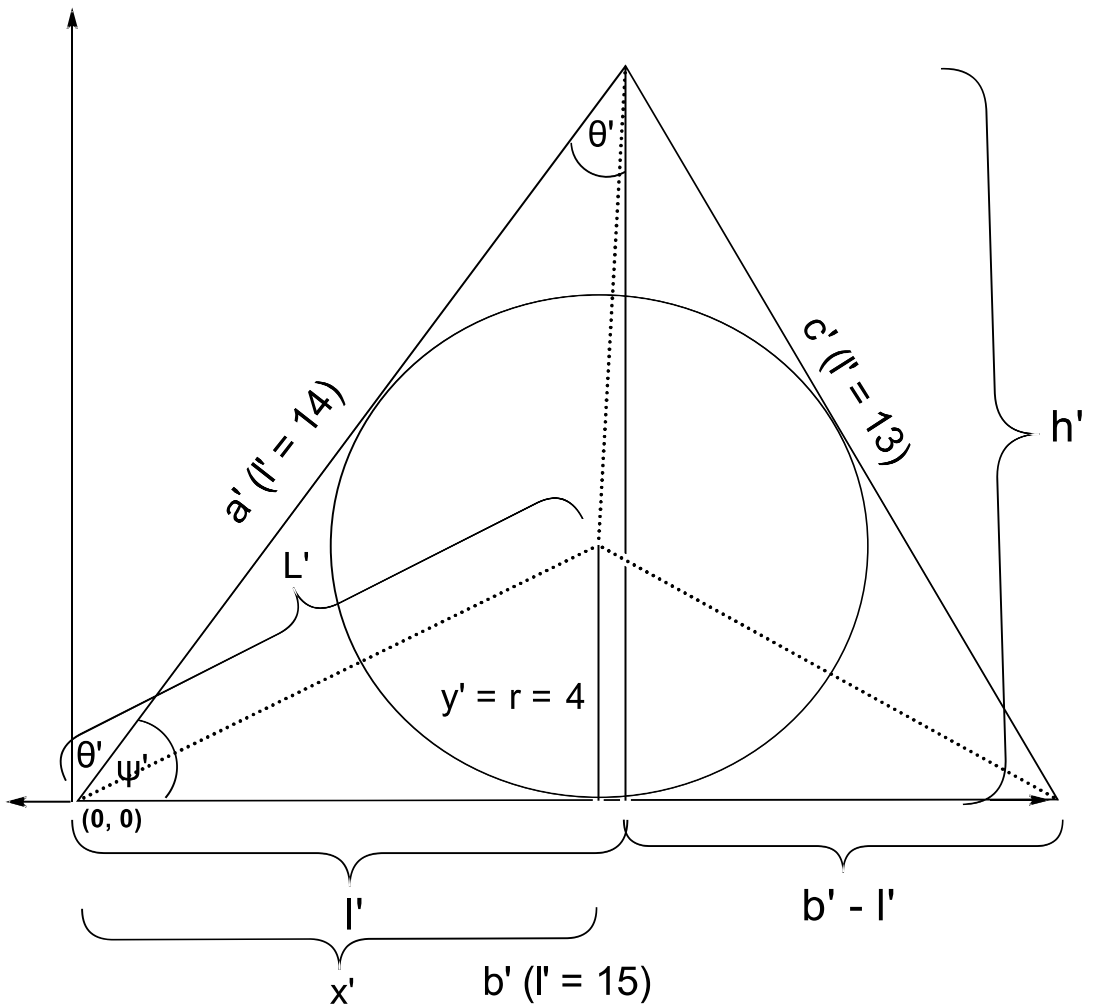
$$\sin\left(\frac{\psi}{2}\right) = \sqrt{\frac{1 - cos(\psi)}{2}} = \sqrt{\frac{1 - \frac{5}{13}}{2}} = 0.55470$$

$$cos\left(\frac{\psi}{2}\right) = \sqrt{\frac{1 + cos(\psi)}{2}} = \sqrt{\frac{1 + \frac{5}{13}}{2}} = 0.83205$$

$$L = \frac{r}{\sin\left(\frac{\psi}{2}\right)} = \frac{4}{\sin\left(\frac{\psi}{2}\right)} = \frac{4}{0.55470} = 7.2111$$

$$x = Lcos\left(\frac{\psi}{2}\right) = (7.2111)(0.83205) = 6 \qquad y = 0$$

To find the next tangent point (c-side):



$$l' = \frac{{a'}^2 + {b'}^2 - {c'}^2}{2b'} = 8.4 \qquad h' = \sqrt{a'^2 - \left(\frac{a'^2 + b'^2 - c'^2}{2b'}\right)^2} = 11.2$$

$$\sin(\psi') = \frac{h'}{a'} = \frac{11.2}{14} \qquad \cos(\psi') = \frac{l'}{a'} = \frac{8.4}{14}$$

$$\sin\left(\frac{\psi'}{2}\right) = \frac{r}{L'} = \frac{4}{L'} \qquad cos\left(\frac{\psi'}{2}\right) = \frac{x}{L'}$$

$$\sin\left(\frac{\psi'}{2}\right) = \sqrt{\frac{1 - cos(\psi')}{2}} = \sqrt{\frac{1 - \frac{8.4}{14}}{2}} = \sqrt{0.2} = 0.447214$$

$$cos\left(\frac{\psi'}{2}\right) = \sqrt{\frac{1 + cos(\psi')}{2}} = \sqrt{\frac{1 + \frac{8.4}{14}}{2}} = \sqrt{0.8} = 0.894427$$

$$L' = \frac{r}{\sin\left(\frac{\psi'}{2}\right)} = \frac{4}{\sin\left(\frac{\psi'}{2}\right)} = \frac{4}{\sqrt{0.2}} = 8.944271$$

$$x' = L'cos\left(\frac{\psi'}{2}\right) = (8.944271)\sqrt{0.8} = 8 \qquad\qquad y' = 0$$

Next, we rotate the triangle by θ' + π/2 counter clockwise followed by translation of the triangle by +14 (b = 14) to get back to the original coordinate system.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta' + \frac{\pi}{2}) & -\sin(\theta' + \frac{\pi}{2}) \\ \sin(\theta' + \frac{\pi}{2}) & \cos(\theta' + \frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 14 \\ 0 \end{bmatrix}$$

$$\theta' = arcsin\left(\frac{8.4}{14}\right) = arccos\left(\frac{11.2}{14}\right) = 0.6435011$$

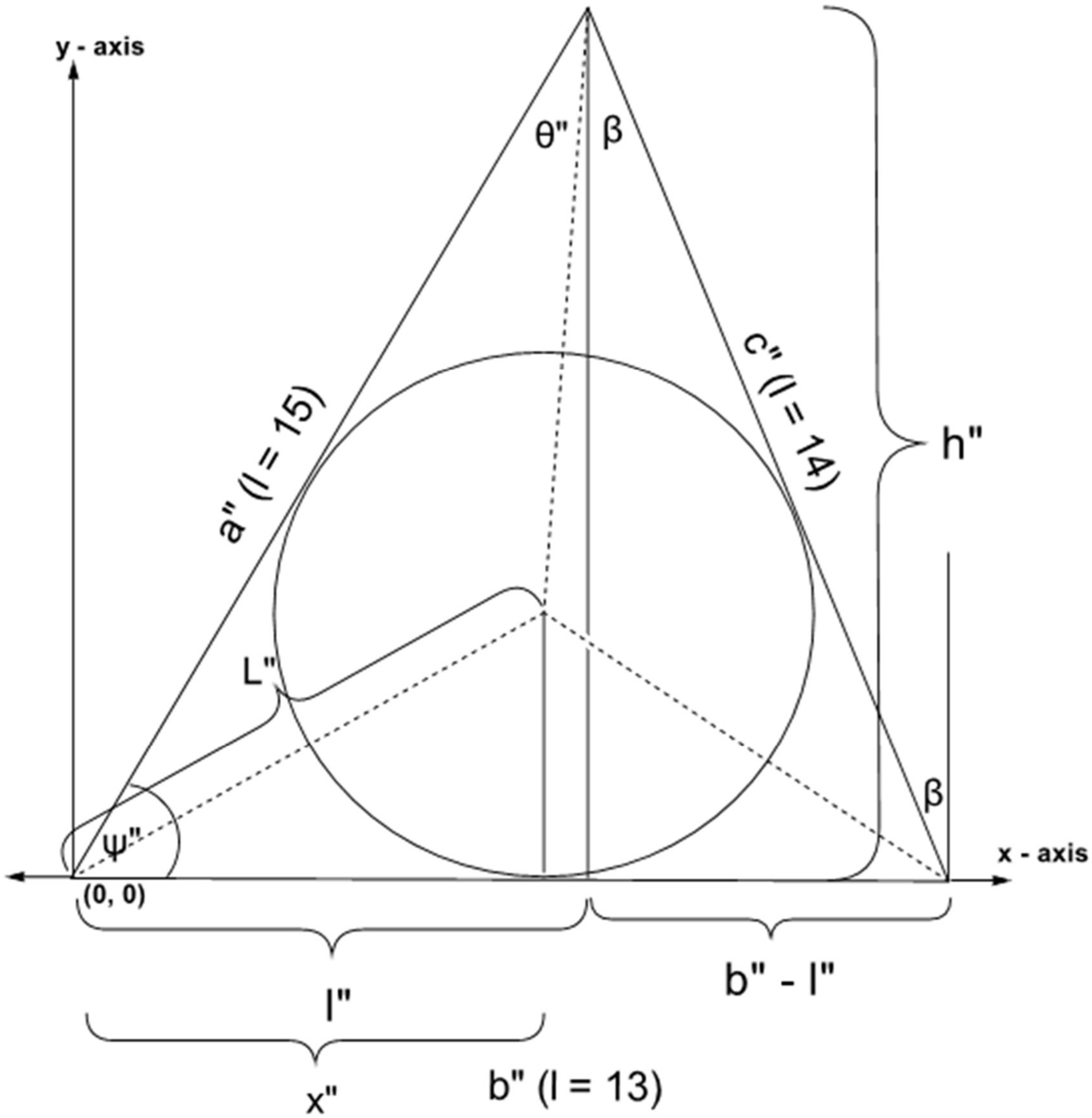
$$\begin{bmatrix} \cos(\theta' + \frac{\pi}{2}) & -\sin(\theta' + \frac{\pi}{2}) \\ \sin(\theta' + \frac{\pi}{2}) & \cos(\theta' + \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \cos(2.214297) & -\sin(2.214297) \\ \sin(2.214297) & \cos(2.214297) \end{bmatrix}$$

$$\begin{bmatrix} \cos(2.214297) & -\sin(2.214297) \\ \sin(2.214297) & \cos(2.214297) \end{bmatrix} \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}_{rotated} = \begin{bmatrix} -4.8 \\ 6.4 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix}_{rotated\ \&\ translated} = \begin{bmatrix} x \\ y \end{bmatrix}_{c-side\ tangent\ point} = \begin{bmatrix} -4.8 + 14 \\ 6.4 \end{bmatrix} = \begin{bmatrix} 9.2 \\ 6.4 \end{bmatrix}$$

Finally, we calculate the a-side tangent point. Referring to the diagram previously used for the a-side excircle:



$$l" = \frac{a"^2 + b"^2 - c"^2}{2b"} = 7.615385 \qquad h" = \sqrt{a"^2 - \left(\frac{a"^2 + b"^2 - c"^2}{2b"}\right)^2} = 12.923077$$

$$\sin(\psi") = \frac{h"}{a"} = \frac{12.923077}{15} \qquad \cos(\psi") = \frac{l"}{a"} = \frac{7.615385}{15}$$

$$\sin\left(\frac{\psi"}{2}\right) = \frac{r}{L"} = \frac{4}{L"} \qquad cos\left(\frac{\psi"}{2}\right) = \frac{x"}{L"}$$

$$\sin\left(\frac{\psi"}{2}\right) = \sqrt{\frac{1 - cos(\psi")}{2}} = \sqrt{\frac{1 - \frac{7.615385}{15}}{2}} = 0.4961389$$

$$cos\left(\frac{\psi''}{2}\right) = \sqrt{\frac{1 + cos(\psi'')}{2}} = \sqrt{\frac{1 + \frac{7.615385}{15}}{2}} = 0.8682431$$

$$L'' = \frac{r}{\sin\left(\frac{\psi''}{2}\right)} = \frac{4}{\sin\left(\frac{\psi''}{2}\right)} = \frac{4}{0.4961389} = 8.062258$$

$$x'' = L''cos\left(\frac{\psi''}{2}\right) = (8.062258)(0.8682431) = 7 \qquad y'' = 0$$

In order to rotate this point (x", y") back into the original coordinate system (x, y), we also need the angle beta. We can find this angle with any inverse trigonometric function, and here we will use the inverse secant function along with a check using the inverse tangent function.

$$sec(\beta) = \frac{c''}{h''} = \frac{14}{12.923077} = 1.083333$$

$$tan(\beta) = \frac{b'' - l''}{h''} = \frac{5.384615}{12.923077} = 0.416667$$

$$\beta = 0.3947914$$

Now we rotate and translate this point (x", y") back into the original coordinate system (x, y) by an initial translation of 13 (length of b") in the negative direction followed by a rotation of the translated point clockwise as a vector through an angle of $\pi/2 + \beta$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\beta + \frac{\pi}{2}) & \sin(\beta + \frac{\pi}{2}) \\ -\sin(\beta + \frac{\pi}{2}) & \cos(\beta + \frac{\pi}{2}) \end{bmatrix} \begin{bmatrix} x'' - 13 \\ y'' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(1.965588) & \sin(1.965588) \\ -\sin(1.965588) & \cos(1.965588) \end{bmatrix} \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.384615 & 0.92307671 \\ -0.92307671 & -0.384615 \end{bmatrix} \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_{a-sid \quad incircle\ tangent\ point} = \begin{bmatrix} 2.30769 \\ 5.53846 \end{bmatrix}$$

a-side tangent = (2.30769, 5.53846); b-side tangent = (6, 0); c-side tangent = (9.2, 6.4)

Appendix I: Tangent points & merged circles

The tangent points of the incircle of a triangle coincide with the tangent points of three circles pushed together until they just touch wherein the radii of the three circles are determined by the lengths of the three sides of the triangle. The circle centers are the vertices of the triangle. The radii of the three touching circles are determined as:

$$R_1, R_2, R_3 \ \textit{are the radii of the three circles} \ \ R_3 > R_2 > R_1$$

$$R_1 + R_2 = a\ (shortest\ side)$$
$$R_1 + R_3 = b\ (medium\ side)$$
$$R_2 + R_3 = c\ (longest\ side)$$

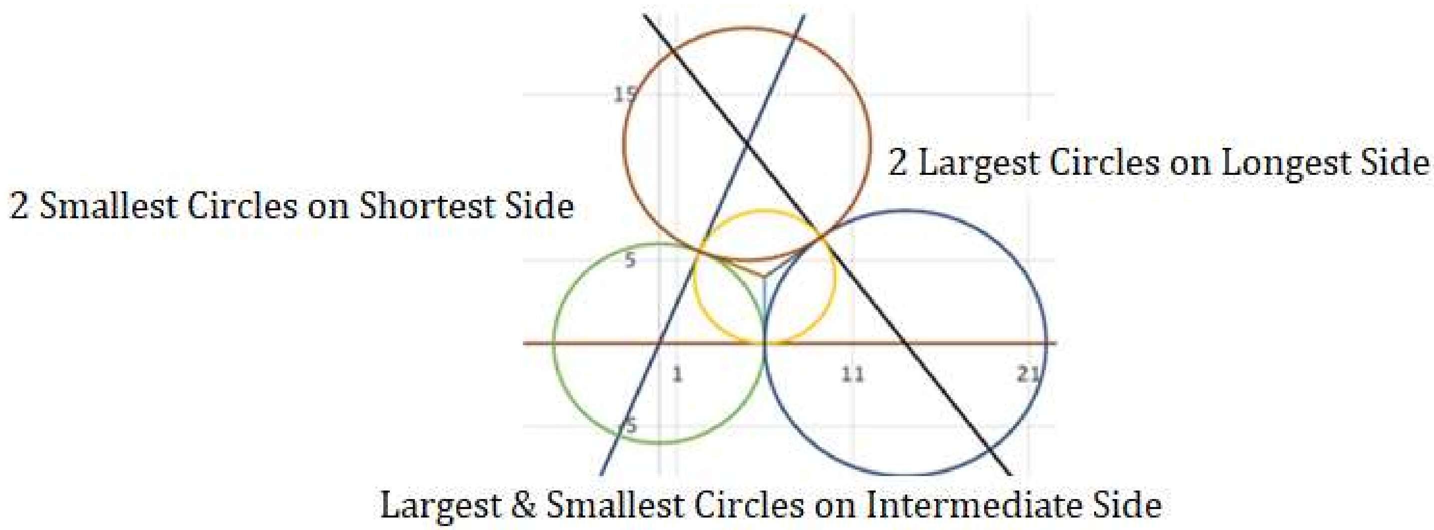
$$R_3 - R_2 = b - a$$

$$2R_3 = b - a + c$$

$$R_3 = \frac{b-a+c}{2} = \frac{c+b-a}{2} \qquad (vertex\ b, c = 0, b)$$

$$R_2 = \frac{b-a+c}{2} = \frac{c+a-b}{2} \qquad (vertex\ c, a = l, h)$$

$$R_1 = \frac{b-a+c}{2} = \frac{a+b-c}{2} \qquad (vertex\ a, b = 0, 0)$$



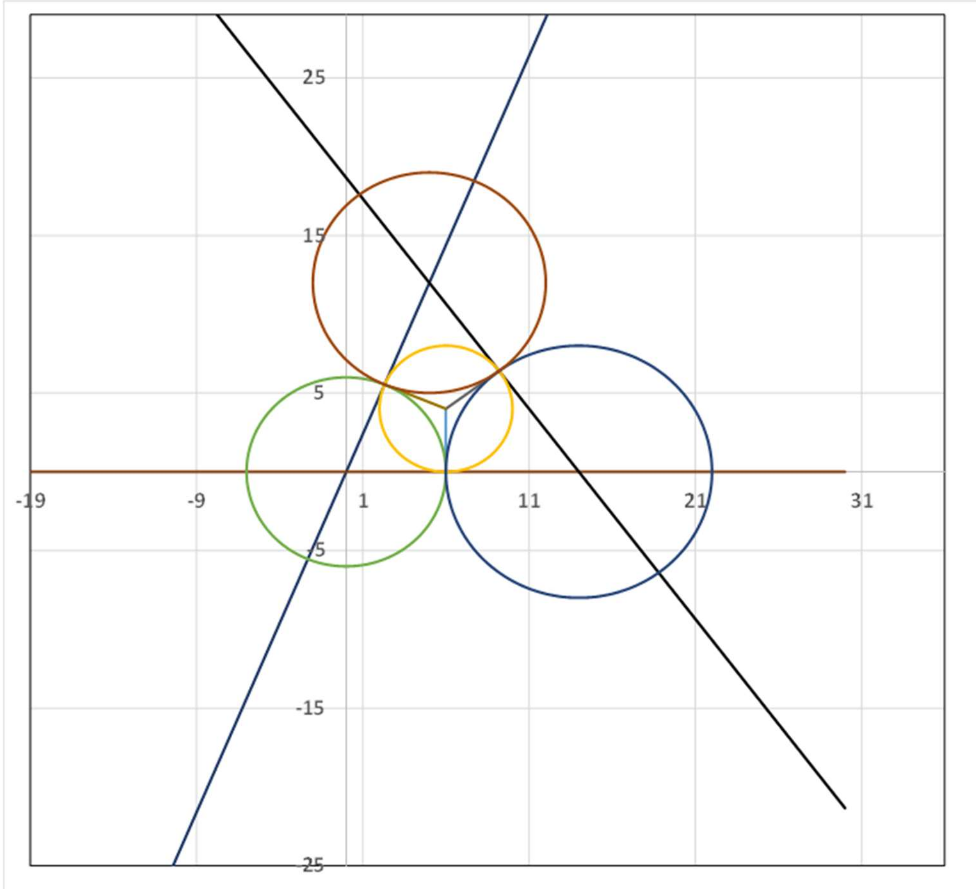
The tangent circles for a 13, 14, 15 triangle can be visualized in the graphic below:

$$R_{incircle} = \frac{A}{s} = \frac{84}{(15+14+13)/2} = 4$$

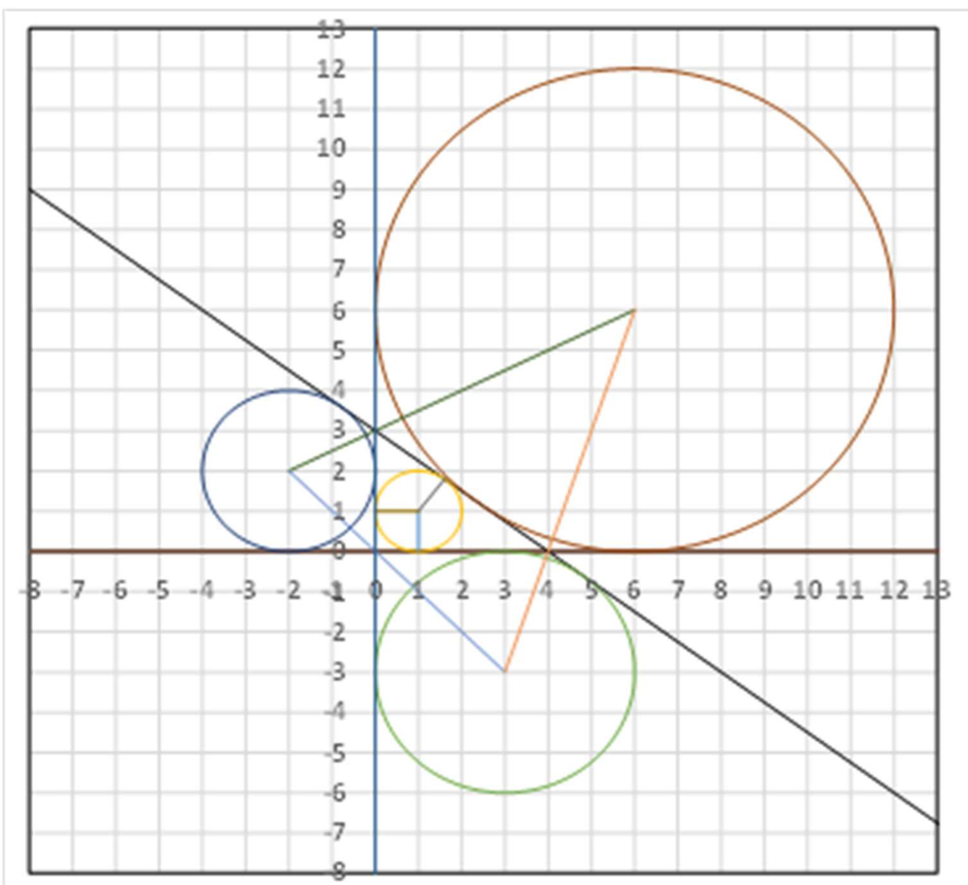
$$R_3 = \frac{c+b-a}{2} = \frac{15+14-13}{2} = 8 \qquad (vertex\ 0, 14)$$

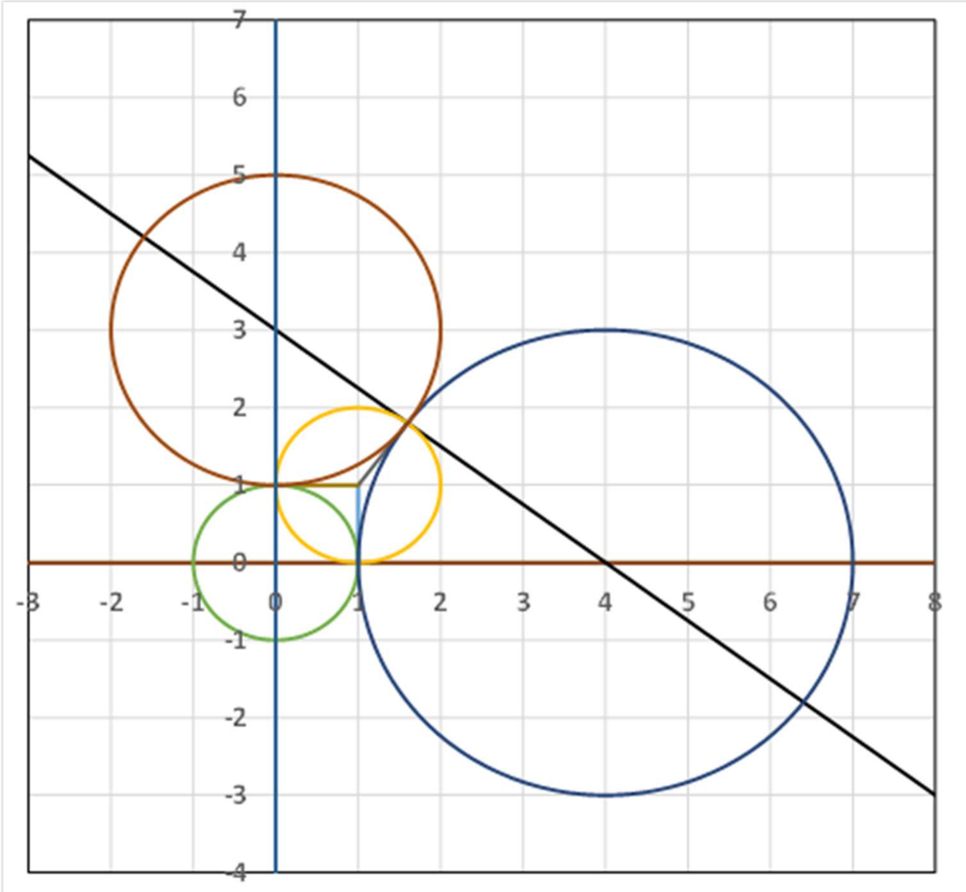
$$R_2 = \frac{c+a-b}{2} = \frac{15+13-14}{2} = 7 \qquad (vertex\ 5, 12)$$

$$R_1 = \frac{a+b-c}{2} = \frac{14+13-15}{2} = 6 \qquad (vertex\ 0, 0)$$

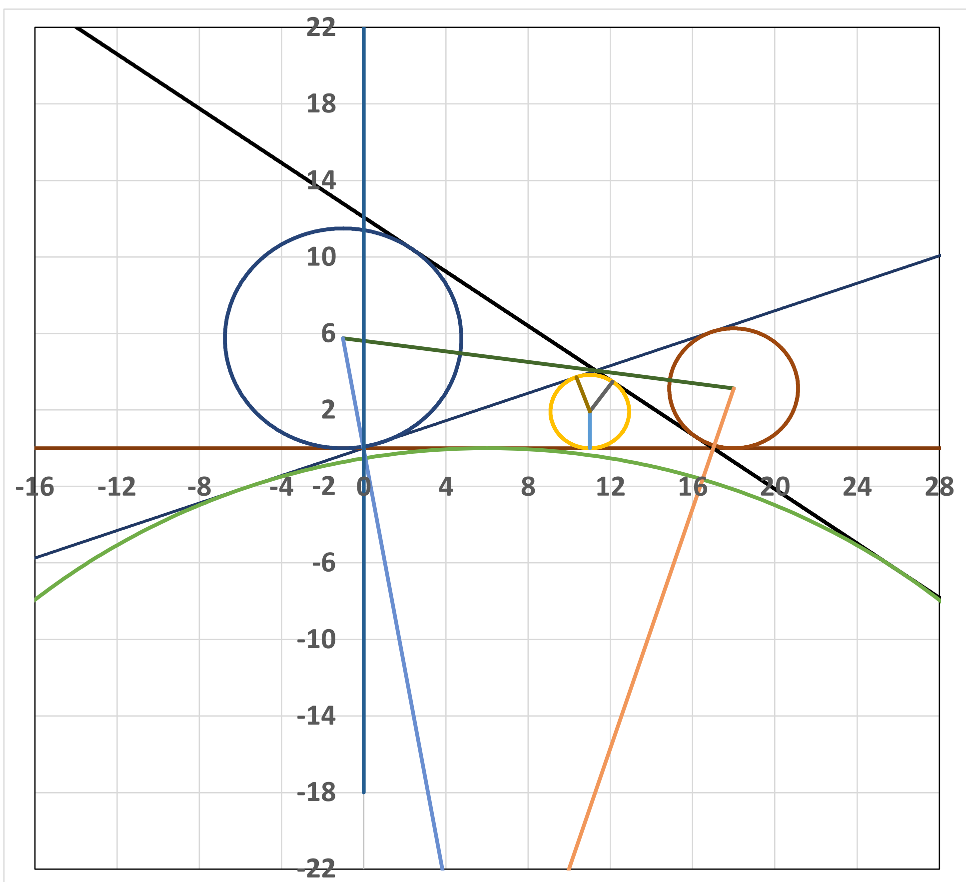


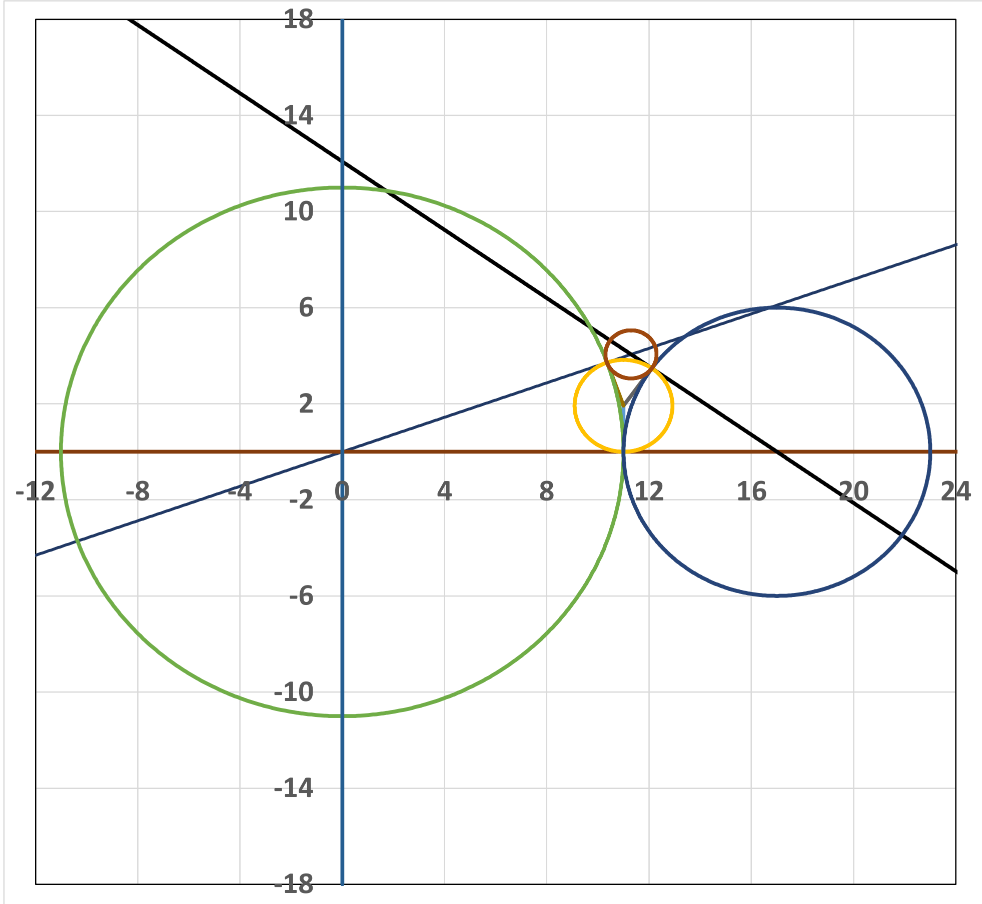
The first graphic below shows the excircles (radii = 6, 3, 2) for a 3, 4, 5 right triangle. The second graphic below shows the merged circles for a 3, 4, 5 triangle (radii = 3, 2, 1). The vertices of the triangle, which are also the centers of the merged circles, are (0, 0), (b, 0), and (l, h); *i.e.*, (0, 0), (4, 0), (0, 3).





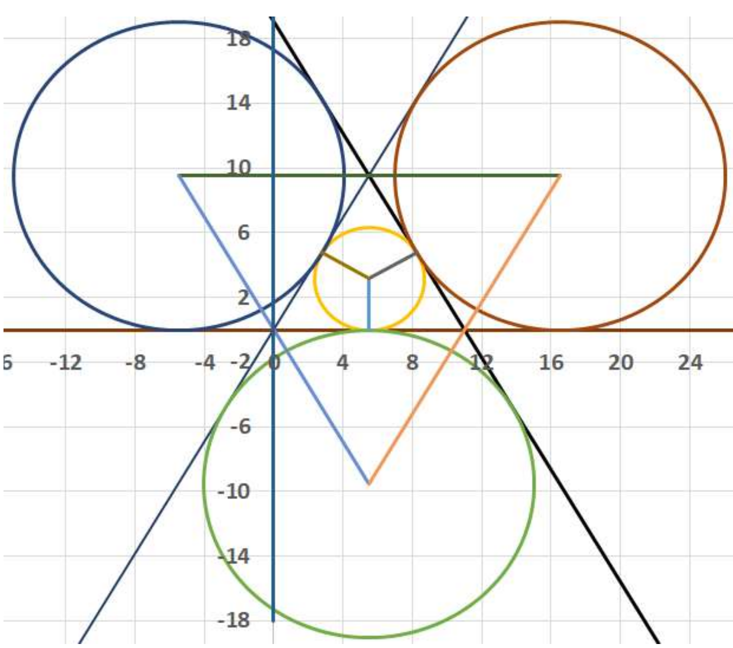
The first graphic below shows the excircles (radii = 34.46737588, 5.744562647, 3.133397807) for a 12, 17, 7 triangle; the second graphic below shows the merged circles for a 12, 17, 7 triangle (radii = 11, 2, 1). The vertices of the triangle (*i.e.*, the centers of the merged circles) are (0, 0), (b, 0), and (l, h); *i.e.*, (0, 0), (17, 0), (11.29411765, 4.054985398).

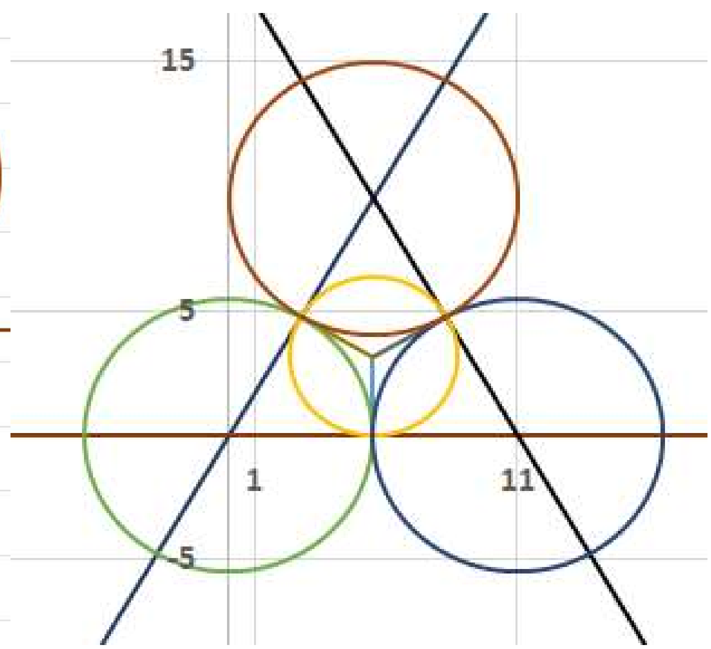




The largest circle is necessarily placed on the vertex between the longest side and the medium side. The smallest circle is placed on the vertex between the shortest side and the medium side. The medium circle is placed on the vertex between the longest side and the shortest side.

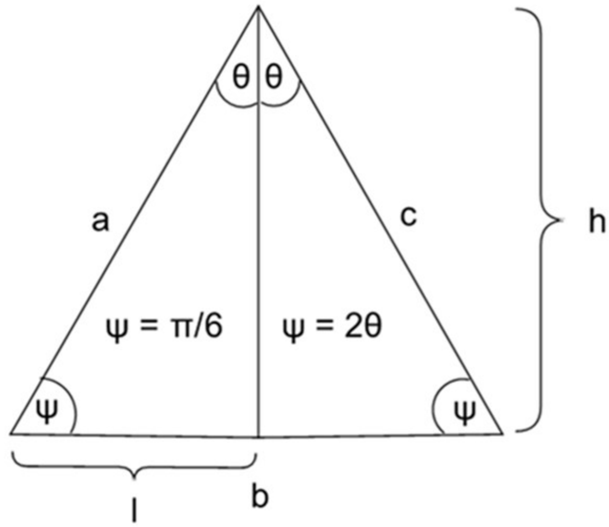
Here is the triangle graphic and the merged circle graphic for an 11, 11, 11 equilateral triangle. The excircle radii are all the same at r = 9.526279442, and the merged circle radii are all the same at r = 5.5; triangle vertices are (0, 0), (11, 0), (5.5, 9.526279442); note that, for an equilateral triangle, the radii of the excircles are equal to the height of the triangle and the radii of the merged circles are equal to the length from the origin to the base of the height line in the triangle (*i.e.*, ½ of the length of a side for an equilateral triangle).





Appendix II: New Triangles with Sides Equal to the Excircle Radii of a Different Triangle:

The excircle radii of a given triangle can be used to create a new triangle ($a' = r_a$; $b' = r_b$; $c' = r_c$). We will refer to this process as triangle iteration. An equilateral triangle has three equal excircle radii, and equilateral triangles "produce" viable new triangles with ever decreasing size forever (*i.e.*, triangle iteration continues ad infinitum).

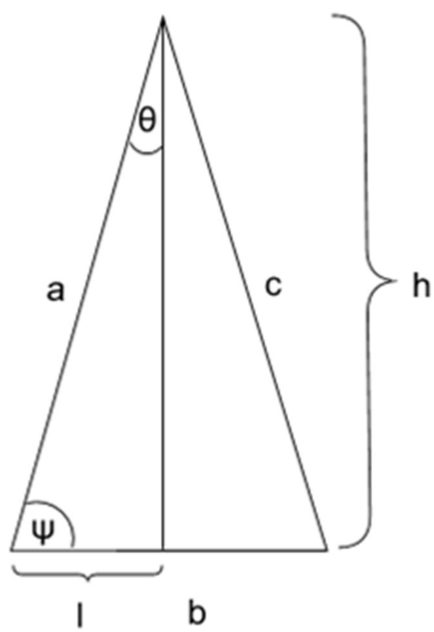


$$r_a = r_b = r_c = \frac{A}{(s-a)} = \frac{A}{(s-b)} = \frac{A}{(s-c)} = \frac{bh/2}{(a+a+a-2a)/2}$$

$$r_a = r_b = r_c == \frac{(b)(a)\sin(\psi)}{a} = (b)\sin\left(\frac{\pi}{6}\right) = (a)\sin\left(\frac{\pi}{6}\right) = (c)\sin\left(\frac{\pi}{6}\right)$$

$$\sin\left(\frac{\pi}{6}\right) = 0.866025 < 1$$

The new values of a', b', and c' after iteration of an equilateral triangle are about 87% of the length of the original a, b, and c; as the new triangle is still equilateral, the iteration process produces viable triangles *ad infinitum*. Not all triangles can be iterated forever. If the excircle radii reach values where the sum of two radii is less than the other radius, then a new triangle can't be produced. Iterated isosceles triangles, are not guaranteed to produce viable new triangles forever.



$$a = c \qquad h < a = c \qquad 0 < b < 2a = 2c \qquad 0 < \psi < \frac{\pi}{2} \qquad 0 < \sin(\psi) < 1$$

$$r_a = r_c = \frac{A}{(s-a)} = \frac{A}{(s-c)} = \frac{(b)(a)\sin(\psi)}{b+c-a} = \frac{(b)(c)\sin(\psi)}{b+a-c} = \frac{(b)(a)\sin(\psi)}{b}$$

$$r_a = r_c = (a)\sin(\psi) = (c)\sin(\psi) = h \qquad (a)\sin(\psi) < a \qquad (c)\sin(\psi) < c$$

$$a' = c' = r_a = r_c < a = c$$

The length of the two equal sides decreases after each iteration.

$$If\ b < a = c \qquad then \qquad \frac{\pi}{6} < \psi < \frac{\pi}{2} \qquad \& \qquad 0.86603 < \sin(\psi) < 1$$

$$r_b = \frac{(b)(a)\sin(\psi)}{(a+c-b)} = q(a)\sin(\psi) = q(c)\sin(\psi) = qh \qquad q = \frac{b}{a+c-b} < 1$$

$$r_b = \frac{(b)(a)\sin(\psi)}{(a+c-b)} = s(b)\sin(\psi) \qquad s = \frac{a}{a+c-b} < 1 \qquad r_b = b' < b$$

$$b' = r_b = qr_a = qr_c = qa' = qc' = qh$$

$$r_b = b' < r_a = r_c = h = a' = c' < a = c$$

An a, b, c isosceles triangle with a = c and b < a = c can be iterated *ad infinitum*. The new values of a', b', and c' are all less than the previous values, and the new values of a' and c' remain equal. Also, the new value of b' remains less than the new equal values of a' and c'. Under these circumstances, the triangle iteration process can continue *ad infinitum*; though the triangle iterates to a steep A-frame with angle ψ approaching π/2 fairly rapidly. The length of sides a & c approach a constant value while the length of side b approaches zero. See the Table below.

Triangle	a'	b'	c'	ψ (degrees)
10, 7, 10 (ψ = 69.51268489°)	9.367496998	5.044036845	9.367496998	74.38150172
9.367496998, 5.044036845, 9.367496998	9.021608675	3.323750564	9.021608675	79.38489215
9.021608675, 3.323750564, 9.021608675	8.867220175	2.002275523	8.867220175	83.51729895
8.867220175, 2.002275523, 8.867220175	8.810523072	1.1213393	8.810523072	86.35143958
8.810523072, 1.1213393, 8.810523072	8.792665486	0.59755979	8.792665486	88.05268116
8.792665486, 0.59755979, 8.792665486	8.787587662	0.309111124	8.787587662	88.99223333
8.787587662, 0.309111124, 8.787587662	8.786228399	0.157298207	8.786228401	89.48711606

$$If\ b > a = c \quad \& \quad b < a + c \quad then \quad 0 < \psi < \frac{\pi}{6} \quad \& \quad 0 < \sin(\psi) < 0.86603$$

$$r_b = \frac{(b)(a)\sin(\psi)}{(a+c-b)} = q(a)\sin(\psi) = q(c)\sin(\psi) = qh \qquad q = \frac{b}{a+c-b} > 1$$

$$b' = r_b > (a)\sin(\psi) = r_a = a' = (c)\sin(\psi) = r_c = c' = h < a = c$$

$$b' = r_b = \frac{(b)(a)\sin(\psi)}{(a+c-b)} = s(b)\sin(\psi) \qquad s = \frac{a}{a+c-b} > 1$$

$$If \quad (s)\sin(\psi) = \left(\frac{a}{a+c-b}\right)\sin(\psi) > 1 \qquad then \qquad b' > b$$

$$(s)\,sin(\psi) = \left(\frac{a}{a+c-b}\right)\left(\frac{h}{a}\right) = \frac{(a)\sin(\psi)}{2a-(2a)\cos(\psi)} = \frac{\sin(\psi)}{2-2\cos(\psi)} = 1$$

$$\sin(\psi) + 2\cos(\psi) = 2 \qquad \psi = 0.927295218 \text{ radians} = 53.13010235°$$

$$(s)\,sin(\psi) > 1 \quad if \quad 0 < \psi < 53.13010235°$$

$$s\,sin(\psi)(55°) = 0.960 \qquad s\,sin(\psi)(45°) = 1.207 \qquad s\,sin(\psi)(30°) = 1.866$$

$$s = \frac{1}{2-2\cos(\psi)} \qquad s(55°) = 1.173 \quad s(45°) = 1.707 \qquad s(30°) = 3.732$$

If b' > b (recall that a' = c' < a = c), then the angle ψ' is less than ψ and the magnitude of the inequality b' > a' = c' is greater than the magnitude of the inequality b > a = c; also, the magnitude of s' will be greater than s leading to the value of b" being greater than b' while a" = c" will be less than a' = c'. With iteration, the length of "b" will continue to increase and the value of "a" & "c" will continue to decrease at a continually increasing rate. If b' > b, then eventually iteration will lead to the length of the longer side of new triangle exceeding the sum of the two shorter sides; and the new triangle will not be viable. Iterated triangles with b' > b diverge quickly (see Table below):

Triangle	a'	b'	c'	ψ (degrees)
7, 10, 7 (ψ = 44.4153086)	4.898979486	12.24744871	4.898979486	Not Viable
11, 12, 11 (ψ = 56.94426885)	9.219544457	11.06345335	9.219544457	53.13010235
9.219544457, 11.06345335, 9.219544457	7.375635565	11.06345335	7.375635565	41.40962209
7.375635565, 11.06345335, 7.375635565	4.878524365	14.63557311	4.878524365	Not Viable
11, 13, 11 (ψ = 53.77845338)	8.874119675	12.81817286	8.874119675	43.76174271
8.874119675, 12.81817286, 8.874119675	6.137883278	15.95849651	6.137883278	Not Viable

$$If \quad (s)\sin(\psi) < 1 \quad then \quad 53.13010235° < \psi < 60° \quad \& \quad b' < b$$

$$If \quad k = \frac{b}{a} \quad \& \quad k' = \frac{b'}{a'} = \frac{s(b)\sin(\psi)}{a\sin(\psi)} = \frac{sb}{a} \quad then \quad k' > k \quad \& \quad b' > a'$$

$$k'' = \frac{b''}{a''} = \frac{s'(b')\sin(\psi')}{a'\sin(\psi')} = \frac{s'b'}{a'} = \frac{(s')(s)(b)\sin(\psi)}{a\sin(\psi)} = \frac{(s')(s)(b)}{a} = s'k'$$

$$\frac{k''}{k'} = \frac{\left(\frac{(s')(s)(b)}{a}\right)}{\left(\frac{sb}{a}\right)} = s' = \frac{a'}{a'+c'-b'} > 1$$

$$\frac{k''}{k'} = s' = \frac{(a)\sin(\psi)}{(a)\sin(\psi)+(c)\sin(\psi)-(s)(b)\sin(\psi)} = \frac{a}{a+c-(s)(b)}$$

$$\frac{k'}{k} = \frac{\left(\frac{sb}{a}\right)}{\left(\frac{b}{a}\right)} = s = \frac{a}{a+c-b} \qquad s > 1 \qquad \frac{k"}{k'} > \frac{k'}{k}$$

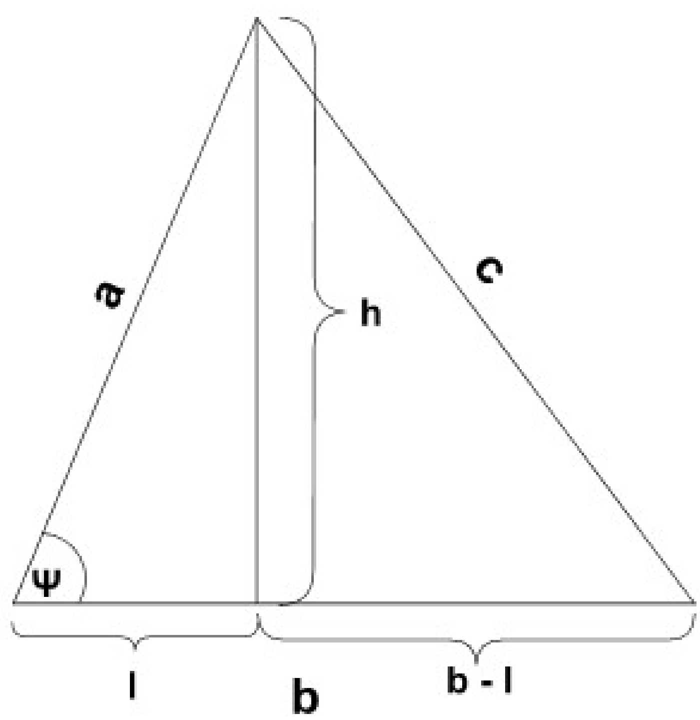
$$\frac{b"}{a"} > \frac{b'}{a'} > \frac{b}{a} > 1 \qquad \psi' < \psi \qquad (s')\sin(\psi') > (s)\sin(\psi)$$

If b > a = c, then the ratio of b to a is greater than one. Even when b' is less than b, the ratio of b' to a' is still greater than the ratio of b to a. With each iteration the magnitude of "b/a" and (s)sin(ψ) increases at a greater rate. Eventually, the magnitude of (s)sin(ψ) increases to a value wherein b' > b. Thus, all isosceles triangles with b > a = c will iterate to a set of "a", "b", and "c" values that do not yield a viable triangle.

The Table below shows the iteration of a 13, 14, 15 scalene triangle until "a" + "b" < "c" (*i.e.*, it is non-viable).

Triangle	Radius Excircle A	Radius Excircle B	Radius Excircle C	ψ (degrees)
13, 14, 15 (ψ = 67.38013505)	10.5	12	14	76.63516676
10.5, 12, 14	7.908881288	9.807012797	14.42207764	108.5181255
7.908881288, 9.807012797, 14.42207764	4.506471913	5.872475287	22.3286772	Not Viable

Scalene triangles always iterate to a set of excircle radii that can't form a viable triangle. See the proof below.



For a scalene triangle with a < b < c & a + b > c:

$$r_a = \frac{A}{(s-a)} = \frac{hb}{b+c-a} = \frac{(a)(b)\sin(\psi)}{b+c-a} < \frac{(a)(b)\sin(\psi)}{c} < \frac{(a)(b)\sin(\psi)}{b}$$

$$r_b = \frac{A}{(s-b)} = \frac{hb}{a+c-b} = \frac{(a)(b)\sin(\psi)}{a+c-b}$$

$$\frac{(a)(b)\sin(\psi)}{a+c-b} > \frac{(a)(b)\sin(\psi)}{a} \qquad \frac{(a)(b)\sin(\psi)}{a+c-b} < \frac{(a)(b)\sin(\psi)}{c}$$

$$r_c = \frac{A}{(s-c)} = \frac{hb}{a+b-c} = \frac{(a)(b)\sin(\psi)}{a+b-c} > \frac{(a)(b)\sin(\psi)}{a} > \frac{(a)(b)\sin(\psi)}{b}$$

$$a' < b' < c' \qquad a' < a \qquad b' < b \qquad c' > h$$

The length of c' is not guaranteed to be longer or shorter than c.

$$\frac{c'}{b'} = \left(\frac{\frac{(a)(b)\sin(\psi)}{a+b-c}}{\frac{(a)(b)\sin(\psi)}{a+c-b}}\right) = \left(\frac{a+c-b}{a+b-c}\right) = \left(\frac{c-(b-a)}{b-(c-a)}\right) > \frac{c}{b}$$

$$\frac{c'}{a'} = \left(\frac{\frac{(a)(b)\sin(\psi)}{a+b-c}}{\frac{(a)(b)\sin(\psi)}{b+c-a}}\right) = \left(\frac{b+c-a}{a+b-c}\right) = \left(\frac{c+(b-a)}{a+(b-c)}\right) > \frac{c}{a}$$

$$\frac{b'}{a'} = \left(\frac{\frac{(a)(b)\sin(\psi)}{a+c-b}}{\frac{(a)(b)\sin(\psi)}{b+c-a}}\right) = \left(\frac{b+c-a}{a+c-b}\right) = \left(\frac{b+(c-a)}{a+(c-b)}\right) > \frac{b}{a}$$

The condition wherein the new values of "a", "b", and "c" no longer form a viable triangle are:

$$\frac{(a)(b)\sin(\psi)}{a+b-c} > \frac{(a)(b)\sin(\psi)}{b+c-a} + \frac{(a)(b)\sin(\psi)}{a+c-b}$$

$$\frac{1}{a+b-c} > \frac{1}{b+c-a} + \frac{1}{a+c-b}$$

$$(a+c-b)(b+c-a) > (a+b-c)(a+c-b) + (a+b-c)(b+c-a)$$

$$c^2 - (a^2+b^2) + 2ab > a^2 - b^2 - c^2 + 2cb + b^2 - a^2 - c^2 + 2ca$$

$$c^2 - c^2 + 2ab > -2c^2 + 2cb + 2ca$$

$$2ab > 2ca - 2c^2 + 2cb$$

$$ab > c(a+b-c)$$

$$1 > \frac{c}{a}\frac{(a+b-c)}{b} = \frac{c}{a}\left(\frac{a}{b} + 1 - \frac{c}{b}\right) = \frac{c}{b} + \frac{c}{a}\left(1 - \frac{c}{b}\right)$$

$$\frac{c'}{b'} > \frac{c}{b} \qquad \frac{a'}{b'} < \frac{a}{b} \qquad \left(1 - \frac{c}{b}\right) < 0 \qquad \frac{c}{b} > 1 \qquad Let\ \left(\frac{c}{b} - 1\right) = k$$

$$1 > \frac{c}{b} - k\left(\frac{c}{a}\right) = c\left(\frac{1}{b} - k\left(\frac{1}{a}\right)\right)$$

$$\frac{\frac{1}{a'}}{\frac{1}{b'}} = \frac{b'}{a'} = \frac{b + (c - a)}{a + (c - b)} > \frac{b}{a} = \frac{\frac{1}{a}}{\frac{1}{b}}$$

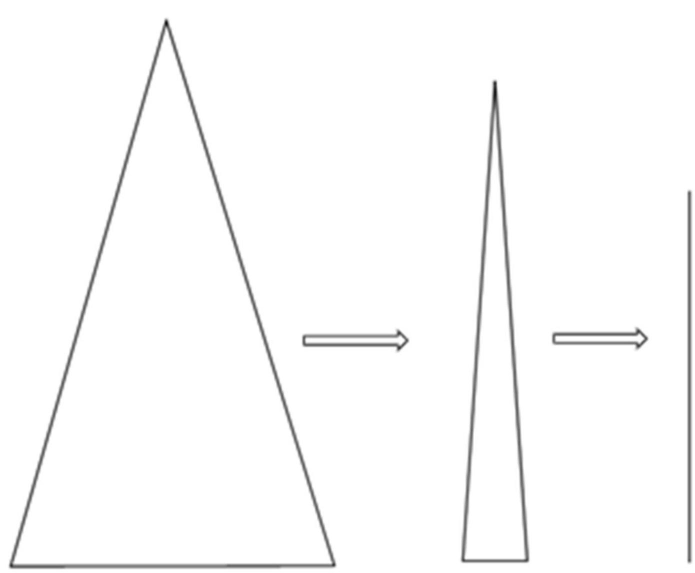
The magnitude of c/b and thus also k increases with each iteration. The magnitude of 1/a and 1/b increases with each iteration, but the value of 1/a increases more rapidly than does 1/b with each iteration.

$$\left(\frac{1}{b} - k\left(\frac{1}{a}\right)\right) > \left(\frac{1}{b'} - k'\left(\frac{1}{a'}\right)\right) \qquad Eventually; \quad 1 > \left(\frac{1}{b^{n'}} - k^{n'}\left(\frac{1}{a^{n'}}\right)\right)$$

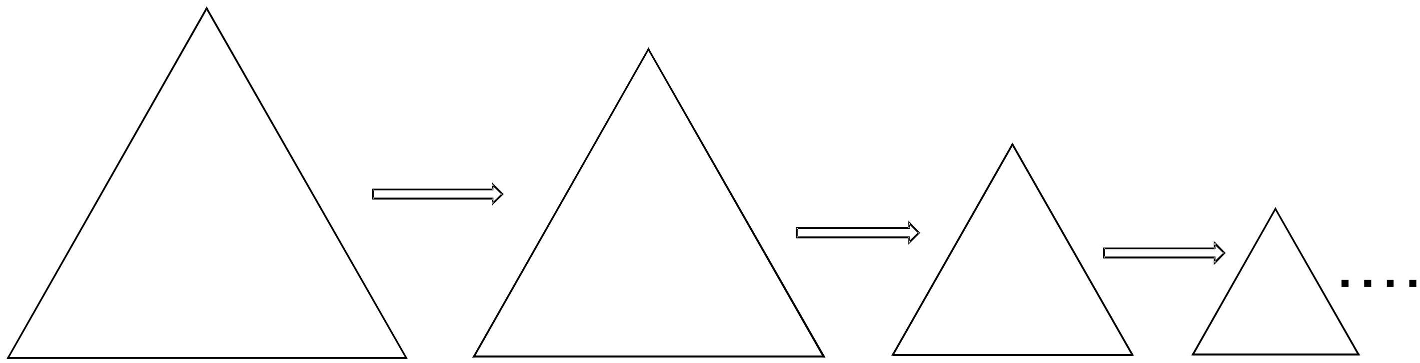
Where superscript n' designates n iterations. Thus, all scalene triangles eventually iterate to a set of excircle radii that do not form a viable triangle. See some examples in the Table below:

$a^{n'}$	$b^{n'}$	$c^{n'}$	$c^{n'}/b^{n'}$	$c^{n'}/a^{n'}$	$b^{n'}/a^{n'}$
4	5	7	1.4	1.75	1.25
2.449489743	3.265986324	9.797958971	Not Viable		
12	14	15	1.071429	1.25	1.166667
9.285504199	12.14258241	14.35032467	1.181818	1.545455	1.307692
6.502677298	9.735646802	15.80926132	1.623853	2.431193	1.497175
1.506764822	2.281448615	66.87171419	Not Viable		
21	22	25	1.136364	1.190476	1.047619
16.80659211	18.20714146	24.27618861	1.333333	1.444444	1.083333
11.90774806	13.36583966	28.47504971	Not Viable		

Isosceles triangles with the two equal sides longer than the third side and equilateral triangles iterate *ad infinitum*. Isosceles triangles with two equal sides longer than third side converge on a line segment.



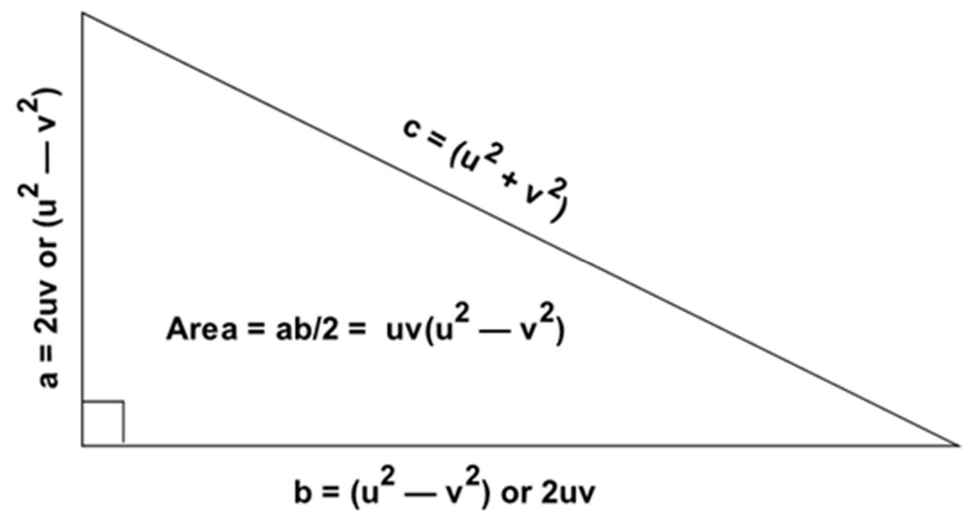
An equilateral triangle iterates to a new equilateral triangle with continuously decreasing side length.



Isosceles triangles with the two equal sides shorter than the third side and scalene triangles iterate eventually to excircle radii that do not form a viable triangle.

Note that a right triangle a, b, c (c > b > a) yields a set of a', b', and c' values that do not yield a viable triangle after one iteration (see proof below Table).

Right Triangle	Radius Excircle A	Radius Excircle B	Radius Excircle C
3, 4, 5	2	3	6
5, 12, 13	3	10	15
20, 21, 29	14	15	35
8, 15, 17	5	12	20
28, 45, 53	18	35	63



$$r_{excircle\ side\ a} = \frac{triangle\ area}{s-a} = \frac{\frac{ab}{2}}{\left(\frac{a+b+c}{2}\right) - \frac{2a}{2}} = \frac{ab}{b+c-a}$$

$$r_{excircle\ side\ b} = \frac{triangle\ area}{s-b} = \frac{\frac{ab}{2}}{\left(\frac{a+b+c}{2}\right) - \frac{2b}{2}} = \frac{ab}{a+c-b}$$

$$r_{excircle\ side\ c} = \frac{triangle\ area}{s-c} = \frac{\frac{ab}{2}}{\left(\frac{a+b+c}{2}\right) - \frac{2c}{2}} = \frac{ab}{a+b-c}$$

$$We\ need\ to\ prove; \qquad c' > a' + b'$$

$$\frac{ab}{a+b-c} > \frac{ab}{b+c-a} + \frac{ab}{a+c-b}$$

$$\frac{1}{a+b-c} > \frac{1}{b+c-a} + \frac{1}{a+c-b}$$

$$(a+c-b)(b+c-a) > (a+b-c)(a+c-b) + (a+b-c)(b+c-a)$$

$$c^2 - (a^2+b^2) + 2ab > a^2 - b^2 - c^2 + 2cb + b^2 - a^2 - c^2 + 2ca$$

$$c^2 - c^2 + 2ab > -2c^2 + 2cb + 2ca$$

$$2ab > 2ca - 2c^2 + 2cb$$

$$ab > c(a+b-c)$$

$$2uv(u^2-v^2) > (u^2+v^2)(2uv + u^2 - v^2 - u^2 - v^2)$$

$$2uv(u^2-v^2) > (u^2+v^2)(2uv - 2v^2)$$

$$2vu^3 - 2uv^3 > 2vu^3 - 2u^2v^2 + 2uv^3 - 2v^4$$

$$0 > 2uv^3 - u^2v^2 - v^4$$

$$0 > uv^2\left(2v - u - \frac{v^2}{u}\right)$$

$$0 > 2v - u - \frac{v^2}{u}$$

$$u + \frac{v^2}{u} > 2v$$

$$\frac{u}{v} + \frac{v}{u} > 2 \qquad u > v$$

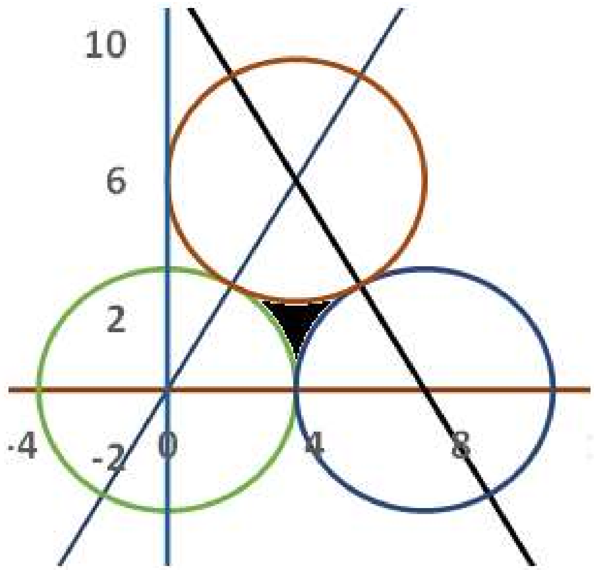
$$\frac{v+a}{v} + \frac{v}{v+a} > 2 \qquad a > 0 \quad v > 0$$

$$(v + a)^2 + v^2 > 2v(v + a)$$

$$2v^2 + 2va + a^2 > 2v^2 + 2va$$

Appendix III: The Largest Triangle Fitted into the Center of Merged Circles:

The challenge here is to find the largest triangle that can be fit into the area between three circles pushed together until they just touch (referred to herein as "merged"). The graphic below shows three merged circles of radius equal to 3.5 arrayed on a 7, 7, 7 equilateral triangle. The total area available for placement of a triangle is shaded black. We can calculate the black shaded area fairly easily (recall that the angles inside an equilateral triangle are all 60°).

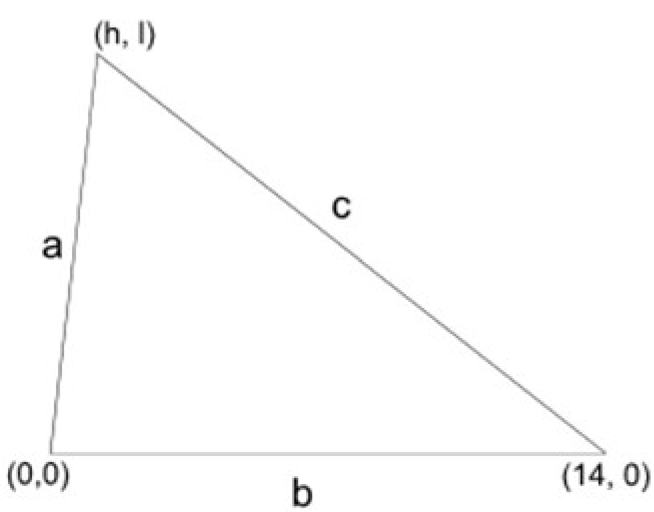


$$Black\ Shaded\ Area = A = Area\ Triangle - Area\ Circles\ Inside\ Triangle$$

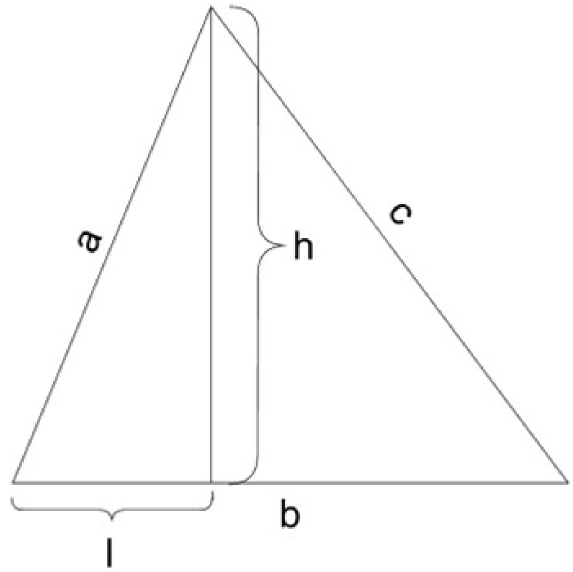
$$A = \frac{hl}{2} - \frac{\pi r^2}{2} = \frac{hb - \pi r^2}{2} = \frac{(6.062177826)(7) - \pi(3.5)^2}{2} = 3.950734776$$

Step 1: Determine the three vertices of the triangle. For illustration, determine the vertices of the triangle associated with merged circles having radii $R_1 = 4$, $R_2 = 6$, and $R_3 = 10$. The length of triangle sides a, b, & c are:

$$R_1 + R_2 = a = 4 + 6 = 10$$
$$R_1 + R_3 = b = 4 + 10 = 14$$
$$R_2 + R_3 = c = 6 + 10 = 16$$



The first vertex for the center of the $R_1 = 4$ circle is placed at the origin (0, 0). The second vertex for the $R_3 = 10$ circle is placed at coordinates (14, 0). To find the coordinates of the third vertex (center of the $R_2 = 6$ circle), one must calculate h & l.

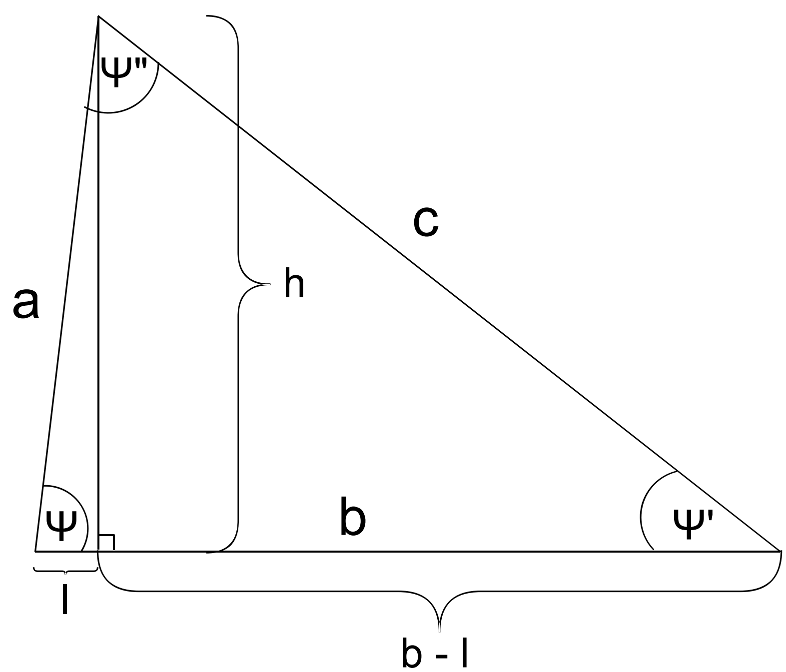


$$l = \frac{a^2 + b^2 - c^2}{2b} = \frac{10^2 + 14^2 - 16^2}{2(14)} = \frac{40}{28} = 1.428571429$$

$$h = \sqrt{a^2 - l^2} = \sqrt{10^2 - \left(\frac{40}{28}\right)^2} = 9.897433186$$

The coordinates of the third vertex are (1.428571429, 9.897433186).

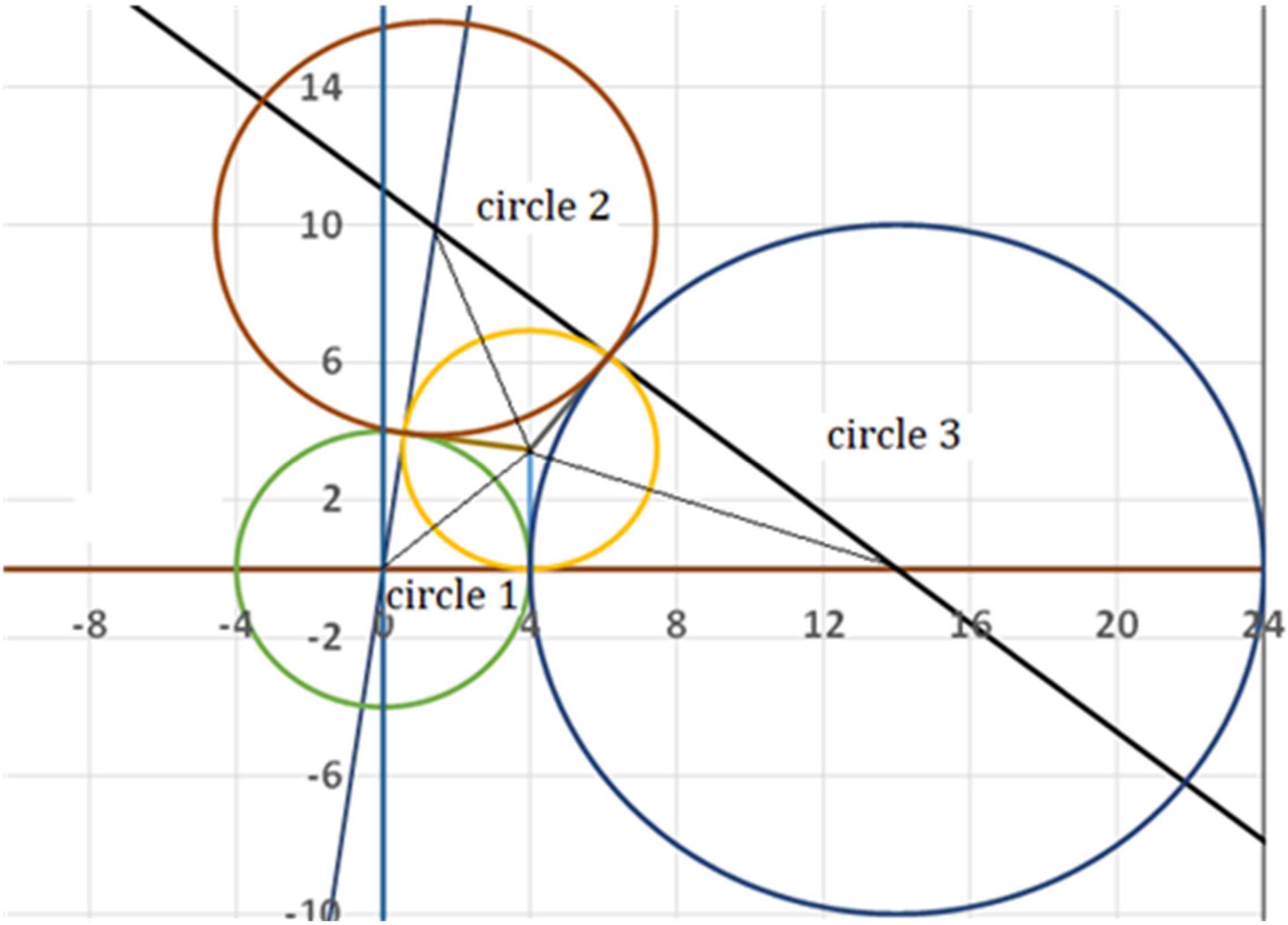
Step 2: Determine the three angles in the triangle.



$$\psi = \arcsin\left(\frac{h}{a}\right) = arcsin\left(\frac{9.897433186}{10}\right) = 81.78678905°$$

$$\psi' = \arcsin\left(\frac{h}{c}\right) = arcsin\left(\frac{9.897433186}{16}\right) = 38.2132107°$$

$$\psi'' = 180° - 81.78678905° - 38.2132107° = 60°$$



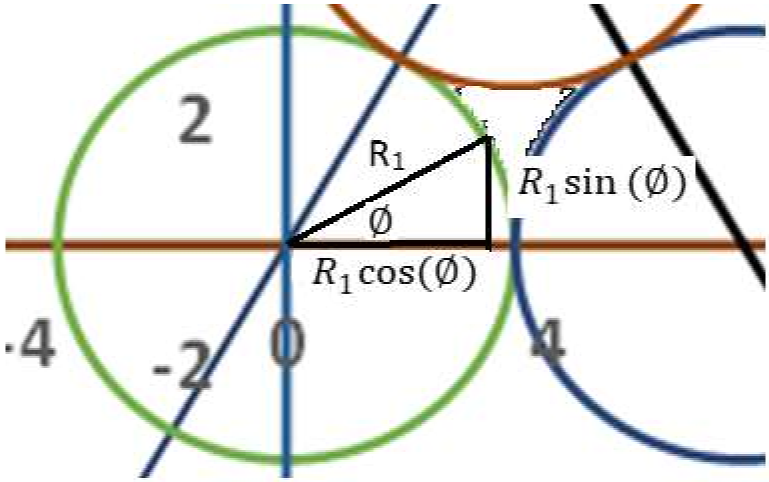
Step 3: Calculate the half angles.

$$\emptyset = \frac{\psi}{2} = 40.89339453°$$

$$\emptyset' = \frac{\psi'}{2} = 19.10660535°$$

$$\emptyset'' = \frac{\psi''}{2} = 30°$$

Step 4: Determine the points of greatest "intrusion" of circles 1, 2, & 3 into the interior of the merged circles with the half angles.



$$x\ coordinate\ of circle\ 1\ intrusion = (R_1)\cos(\emptyset) = 3.02371579$$
$$y\ coordinate\ of\ circle\ 1\ intrusion = (R_1)\sin(\emptyset) = 2.618614677$$
$$x\ coordinate\ circle\ 3\ intrusion = b - (R_3)\cos(\emptyset') = 4.550888175$$
$$y\ coordinate\ circle\ 3\ intrusion = (R_3)\sin(\emptyset') = 3.273268353$$

To find the coordinates of the maximum intrusion of circle 2, one must first rotate the third vertex by the complement of ψ, in this case 8.21321095°, counterclockwise. We know that this will place the vertex at the coordinates (0, a).

$$\begin{bmatrix} x(rotated) \\ y(rotated) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

We next carry out the following calculation:

$$\begin{bmatrix} x(new) \\ y(new) \end{bmatrix} = \begin{bmatrix} 0 + R_2\sin(\psi'') \\ 10 - R_2\cos(\psi'') \end{bmatrix} = \begin{bmatrix} 0 + (6)\sin(30°) \\ 10 - (6)\cos(30°) \end{bmatrix} = \begin{bmatrix} 3 \\ 4.803847577 \end{bmatrix}$$

Rotate the point clockwise by the complement of ψ to get back to the original coordinate system:

$$\begin{bmatrix} x(final) \\ y(final) \end{bmatrix} = \begin{bmatrix} \cos(8.21321095°) & \sin(8.21321095°) \\ -\sin(8.21321095°) & \cos(8.21321095°) \end{bmatrix} \begin{bmatrix} 3 \\ 4.803847577 \end{bmatrix}$$

$$\begin{bmatrix} x(final) \\ y(final) \end{bmatrix} = \begin{bmatrix} 3.655493914 \\ 4.326004599 \end{bmatrix}$$

$$x\ coordinate\ circle\ 2\ intrusion = 3.655493914$$
$$y\ coordinate\ circle\ 2\ intrusion = 4.326004599$$

Step 5: Determine the straight lines which incorporate the line segments that are the sides of the greatest area triangle. The slopes of these lines are the negative inverse of the slopes of the lines from the circle centers to the points of maximum intrusion (one can also use the center of the incircle).

$$Circle\ One\ Line\ Slope = \tan(\emptyset) = \tan(40.89339453°) = 0.8660254$$

$$Circle\ One\ Perpendicular\ Line\ Slope = -1.154700543$$

This line includes the point of maximum intrusion.

$$y = mx + b$$

$$2.618614677 = (-1.154700543)(3.02371579) + b$$

$$b = 2.618614677 + (1.154700543)(3.02371579) = 6.110100942$$

$$\boxed{y = (-1.154700543)x + 6.110100942}$$

$$Circle\ Three\ Line\ Slope = -\tan(\emptyset') = -\tan(19.10660535°) = -0.34641016$$

$$Circle\ Three\ Perpendicular\ Line\ Slope = 2.886751346$$

$$y = mx + b$$

$$3.273268353 = (2.886751346)(4.550888175) + b$$

$$b = 3.273268353 - (2.886751346)(4.550888175) = -9.864014212$$

$$\boxed{y = (2.886751346)x - 9.864014212}$$

The easiest way to calculate this slope is by direct calculation.

$$Circle\ Two\ Line\ Slope = \frac{4.326004599 - 9.897433186}{3.655493914 - 1.428571429} = -2.501851153$$

$$Circle\ Two\ Line\ Perpendicular\ Slope = 0.399704034$$

$$y = mx + b$$

$$3.655493914 = (0.399704034)(4.326004599) + b$$

$$b = 4.326004599 - (0.399704034)(3.655493914) = 2.864888935$$

$$\boxed{y = (0.399704034)x + 2.864888935}$$

Step 6: Calculate the intersections of the straight lines derived in the previous section.

$$y = (-1.154700543)x + 6.110100942$$
$$y = (2.886751346)x - 9.864014212$$
$$(4.041451889)x - 15.97411515 = 0$$
$$\boxed{x = 3.952568432 \qquad y = 1.546068028}$$

$$y = (-1.154700543)x + 6.110100942$$
$$y = (0.399704034)x + 2.864888935$$
$$(1.554404577)x - 3.245212007 = 0$$
$$\boxed{x = 2.087752478 \qquad y = 3.69937204}$$

$$y = (2.886751346)x - 9.864014212$$
$$y = (0.399704034)x + 2.864888935$$
$$(2.487047312)x - 12.72890315 = 0$$
$$\boxed{x = 5.118078408 \qquad y = 4.910605521}$$

Step 7: Determine the length of the triangle sides.

$$length = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

$$length = \sqrt{(1.546068028 - 3.69937204)^2 + (3.952568432 - 2.087752478)^2}$$

$$length = 2.848553453 = a$$

$$length = \sqrt{(1.546068028 - 4.910605521)^2 + (3.952568432 - 5.11807840)^2}$$

$$length = 3.263427955 = b$$

$$length = \sqrt{(3.69937204 - 4.910605521)^2 + (2.087752478 - 5.118078408)^2}$$

$$length = 3.560691817 = c$$

Step 8: Determine the area of the triangle.

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{a+b+c}{2} = 4.836336613$$

$$Area\ Enclosed\ Triangle = 4.391970313$$

Step 9: Determine the total area enclosed within the merged circles.

$$A = \frac{hb}{2} - \left(\frac{\psi}{360}\right)(R_1)^2 - \left(\frac{\psi'}{360}\right)(R_2)^2 - \left(\frac{\psi''}{360}\right)(R_3)^2$$

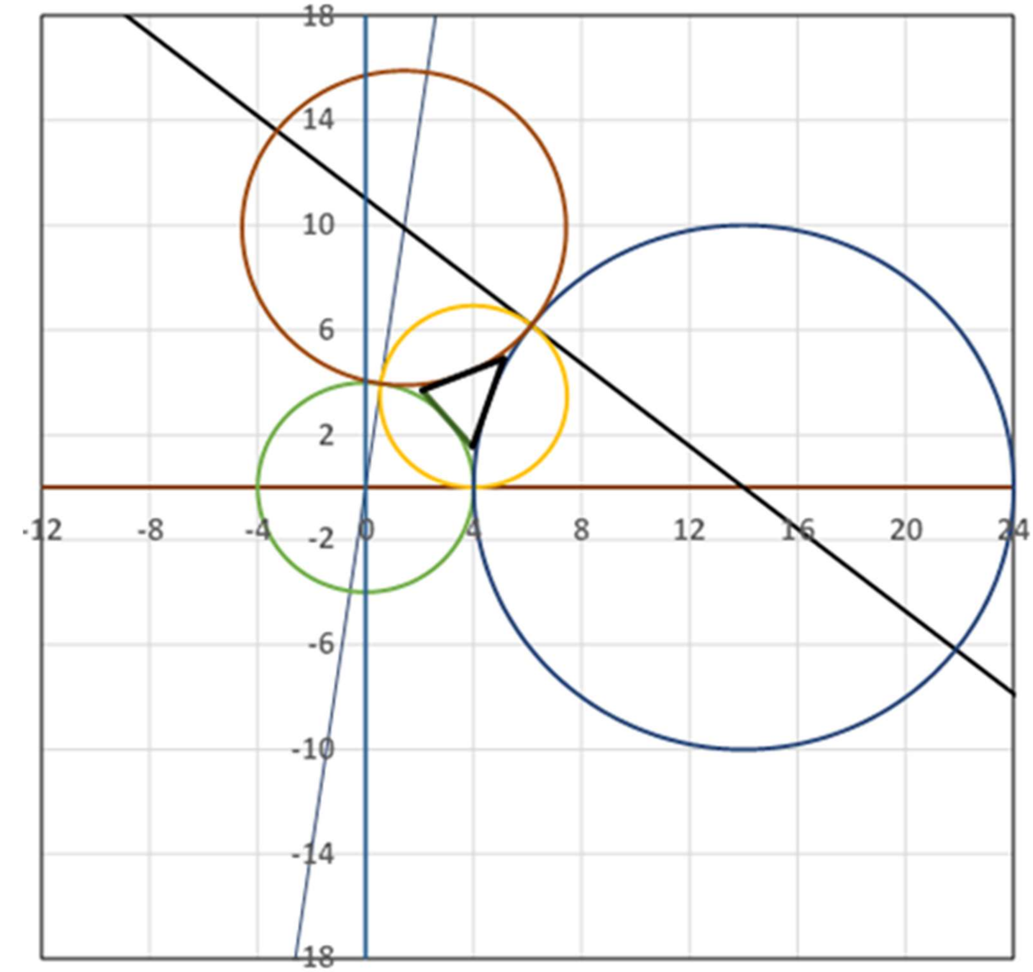
$$A = \frac{(9.89743318)(14)}{2} - \left(\frac{81.7867890°}{360°}\right)\pi(4)^2 - \left(\frac{38.2132107°}{360°}\right)\pi(10)^2 - \left(\frac{60°}{360°}\right)\pi(6)^2$$

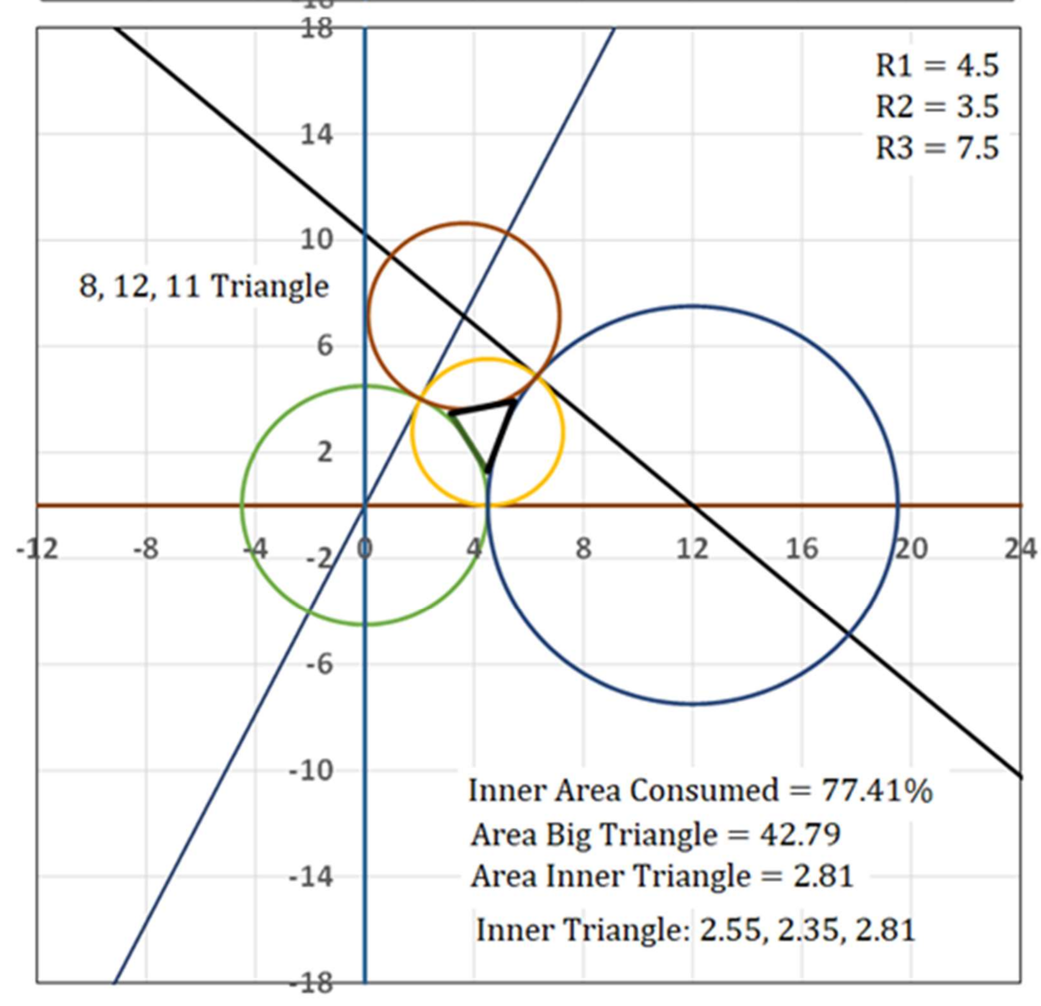
$$A = 69.2820323 - 11.41959003 - 33.34731722 - 18.84955592 = 5.66556913$$

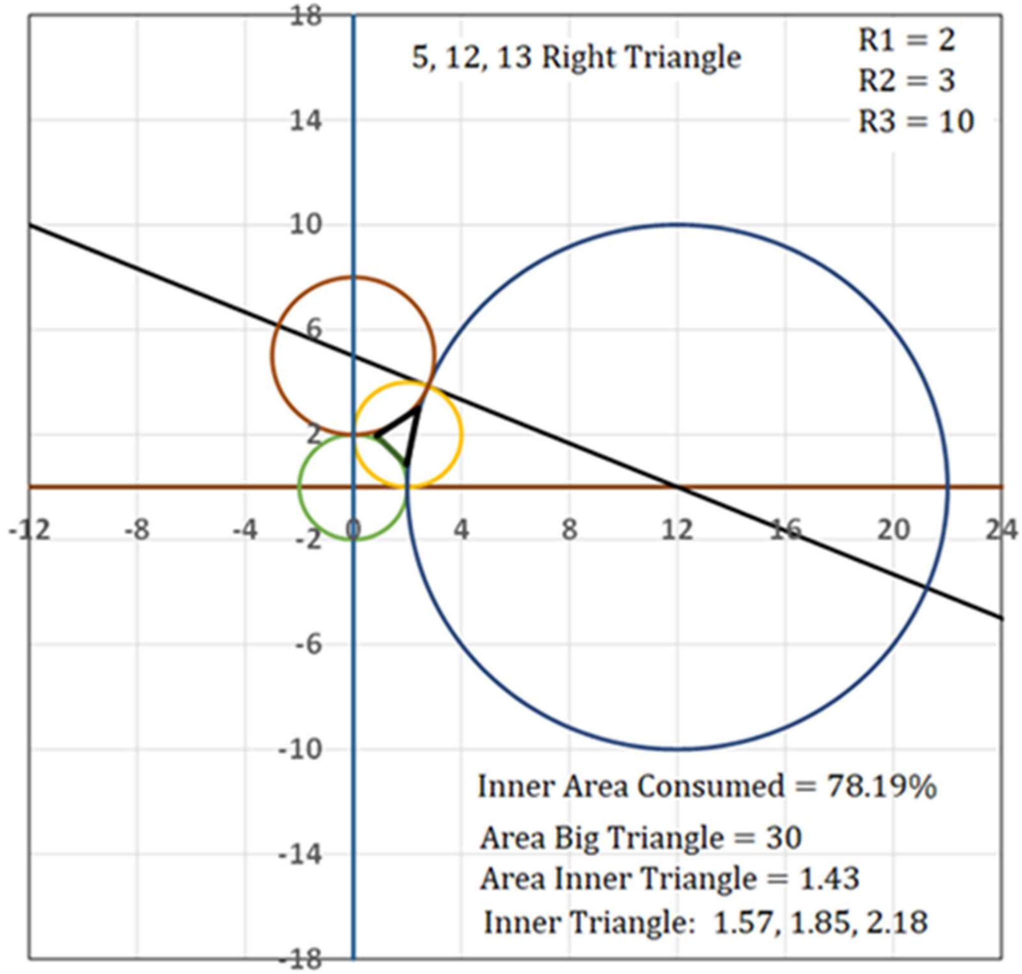
Step 10: Determine the % of available area consumed by the enclosed triangle.

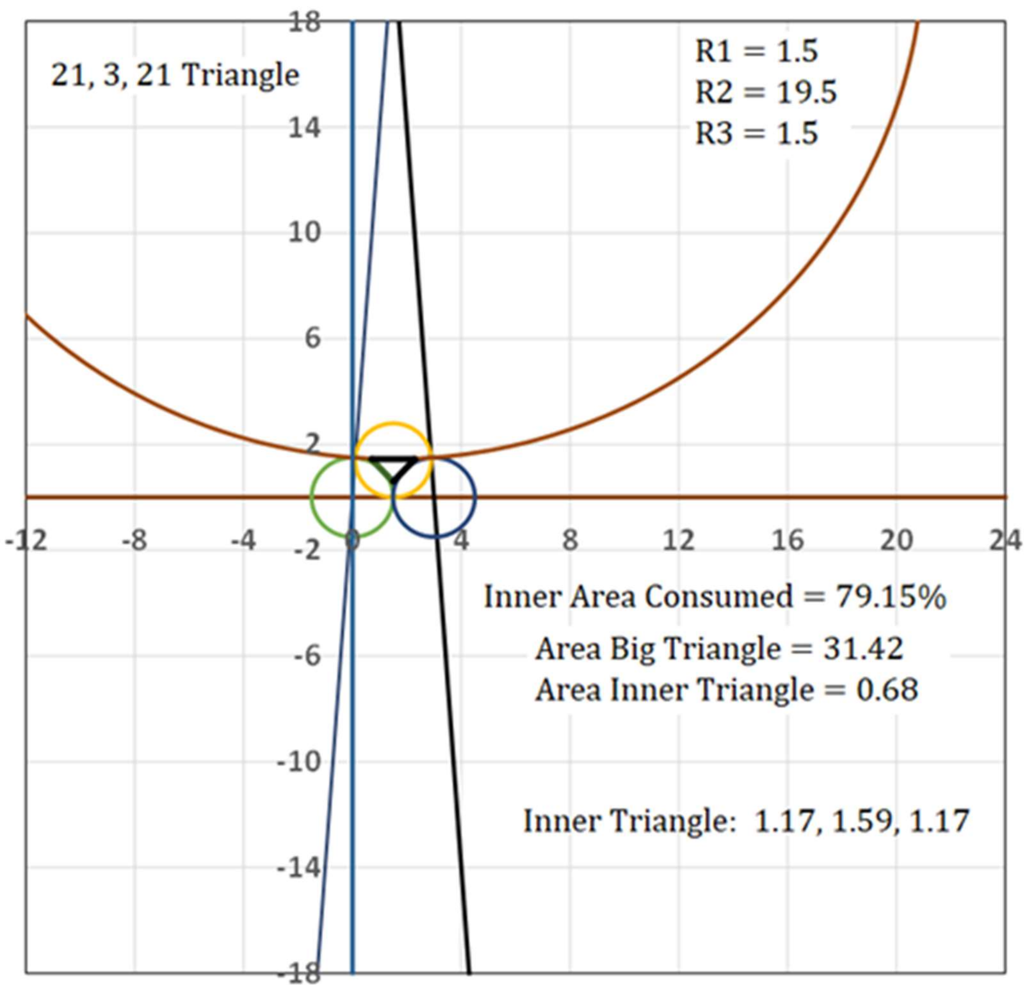
$$\%\ Area\ Consumed = \frac{4.391970313}{5.66556913}(100) = 77.52\%$$

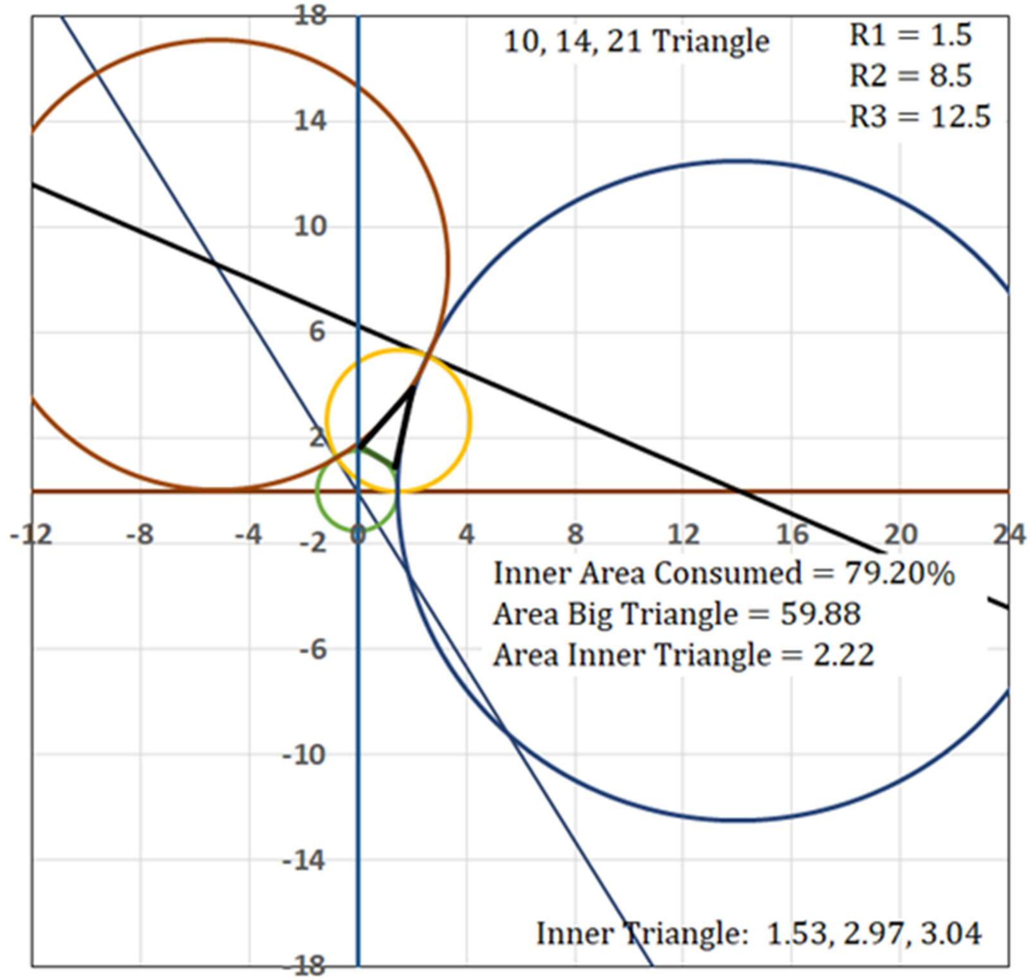
An Excel sheet was composed to illustrate the "greatest area triangle" that fits into the area between merged circles. See the graphic immediately below for three circles of radii = 4, 6, 10 centered on the vertices of a 10, 14, 16 triangle, and see the graphics following the 10, 14, 16 triangle for visualization of the greatest area triangle inserted into a few other merged circle combinations superimposed on triangles.

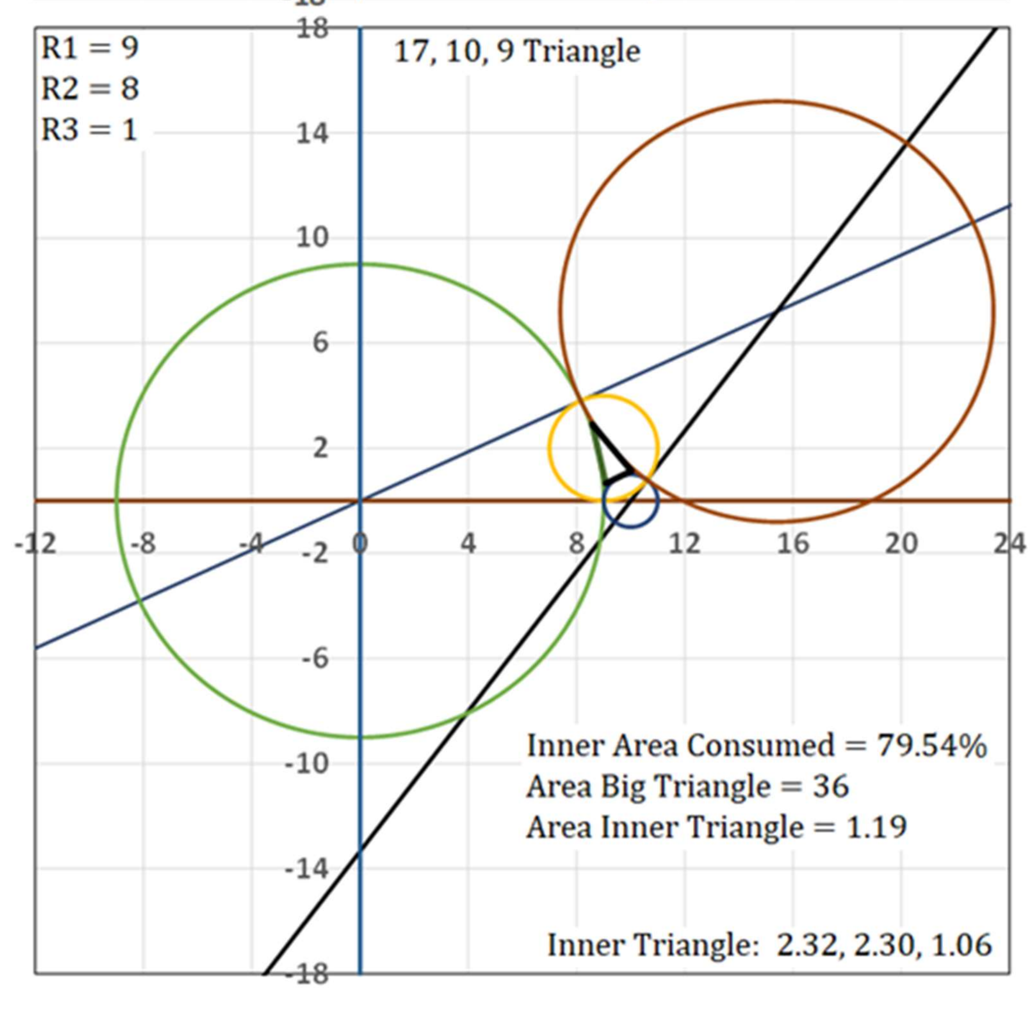

18
14
10
6
2
-2
-6
-10
-14
-18
-12
-8
-4
0
4
8
12
16
20
24


R1 = 4.5
R2 = 3.5
R3 = 7.5
8, 12, 11 Triangle
Inner Area Consumed = 77.41%
Area Big Triangle = 42.79
Area Inner Triangle = 2.81
Inner Triangle: 2.55, 2.35, 2.81
18
14
10
6
2
-2
-6
-10
-14
-18
-12
-8
-4
0
4
8
12
16
20
24

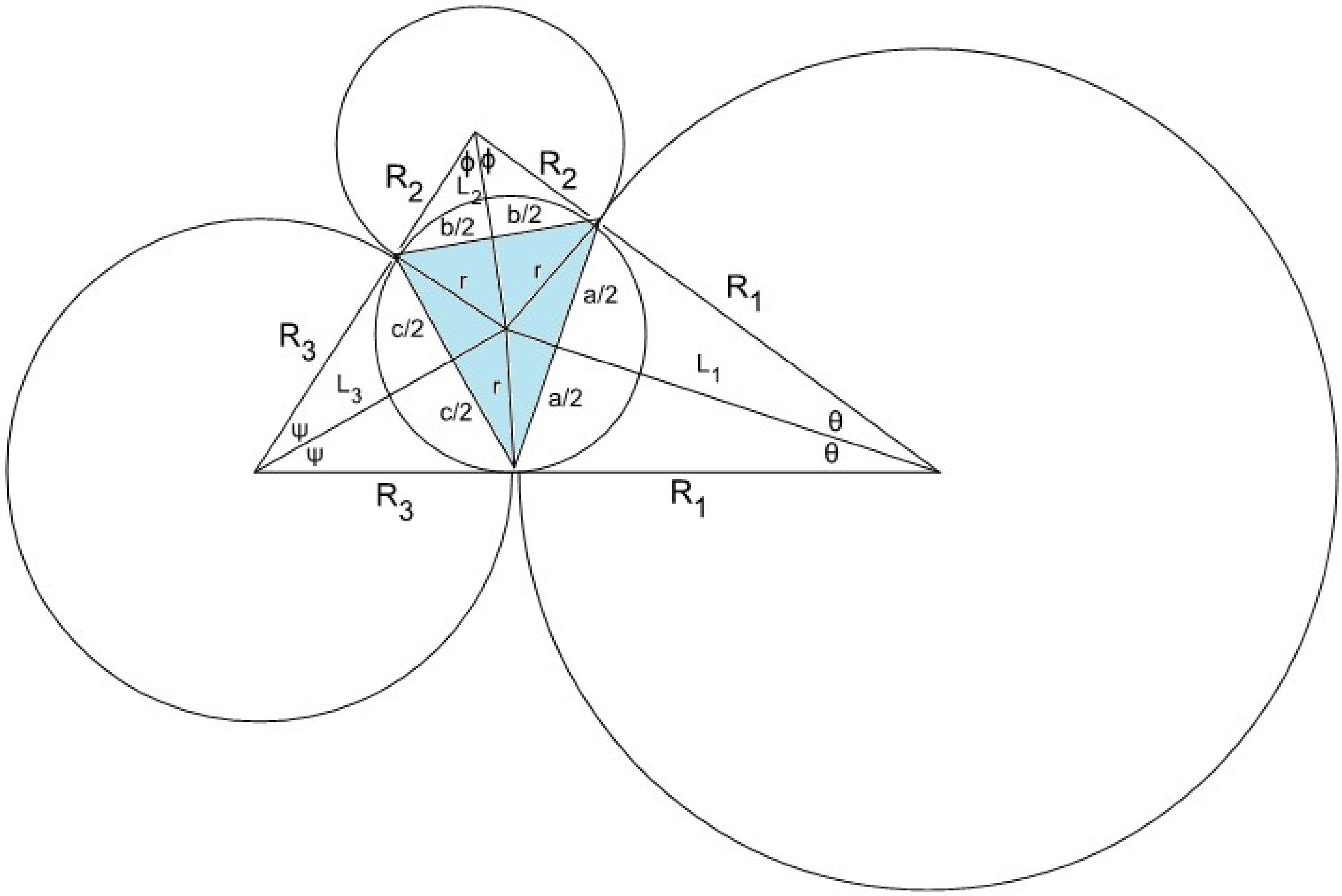

5, 12, 13 Right Triangle
R1 = 2
R2 = 3
R3 = 10
Inner Area Consumed = 78.19%
Area Big Triangle = 30
Area Inner Triangle = 1.43
Inner Triangle: 1.57, 1.85, 2.18


21, 3, 21 Triangle
R1 = 1.5
R2 = 19.5
R3 = 1.5
Inner Area Consumed = 79.15%
Area Big Triangle = 31.42
Area Inner Triangle = 0.68
Inner Triangle: 1.17, 1.59, 1.17


10, 14, 21 Triangle
R1 = 1.5
R2 = 8.5
R3 = 12.5
Inner Area Consumed = 79.20%
Area Big Triangle = 59.88
Area Inner Triangle = 2.22
Inner Triangle: 1.53, 2.97, 3.04


17, 10, 9 Triangle
R1 = 9
R2 = 8
R3 = 1
Inner Area Consumed = 79.54%
Area Big Triangle = 36
Area Inner Triangle = 1.19
Inner Triangle: 2.32, 2.30, 1.06

See the diagram of three merged circles below. Within the outer triangle ($A = R_1 + R_2$, $B = R_1 + R_3$, $C = R_2 + R_3$) whose vertices are the centers of the three merged circles there is another triangle whose vertices are the three points of contact between the merged circles (see the blue shaded area in the diagram). The side lengths of the inner triangle can be calculated by the three equations below the diagram; the inner triangle is a, b, c.

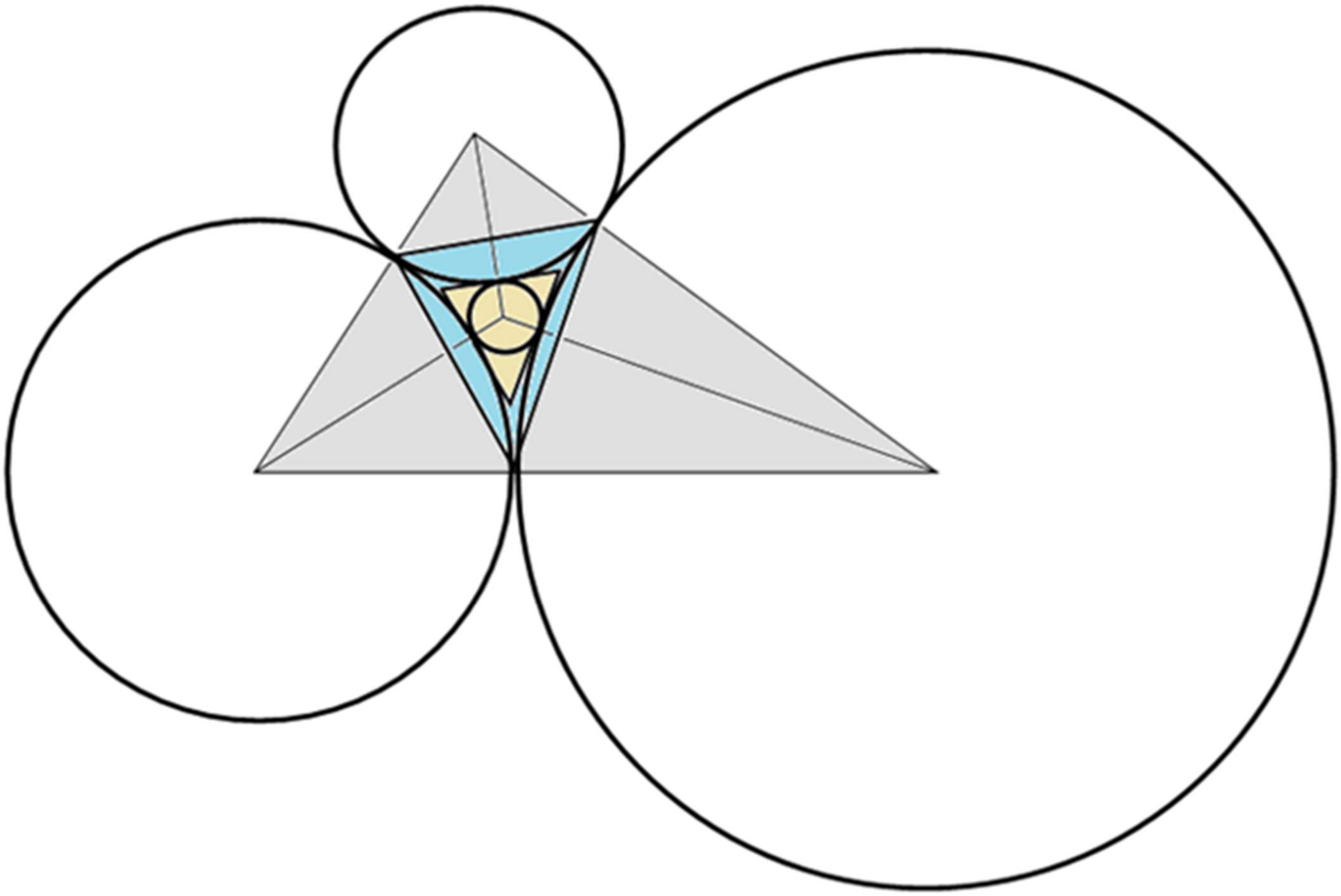


$$\frac{a}{2} = R_1 \sin\left\{\arctan\left(\frac{r}{R_1}\right)\right\} = R_1 \sin(\theta) \qquad L_1 = R_1 \cos\left\{\arctan\left(\frac{r}{R_1}\right)\right\} = R_1 \cos(\theta)$$

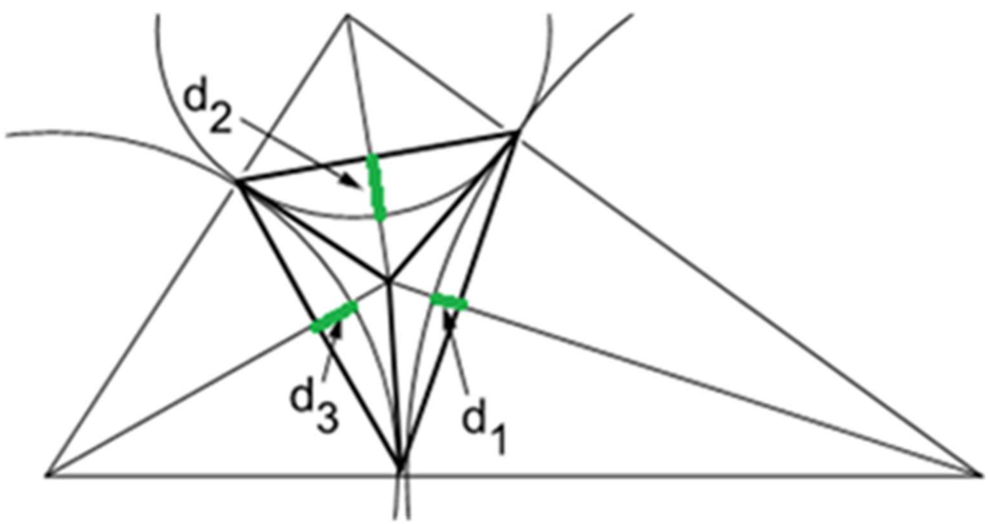
$$\frac{b}{2} = R_2 \sin\left\{\arctan\left(\frac{r}{R_2}\right)\right\} = R_2 \sin(\emptyset) \qquad L_2 = R_2 \cos\left\{\arctan\left(\frac{r}{R_2}\right)\right\} = R_1 \cos(\emptyset)$$

$$\frac{c}{2} = R_3 \sin\left\{\arctan\left(\frac{r}{R_3}\right)\right\} = R_3 \sin(\psi) \qquad L_3 = R_3 \cos\left\{\arctan\left(\frac{r}{R_3}\right)\right\} = R_3 \cos(\psi)$$

Within the inner triangle, there is another triangle that is the greatest area triangle that can fit in the space between the merged circles. See the diagram below for the locations of the main triangle whose vertices are the centers of the merged circles (grey, blue, & tan area), the inner triangle whose vertices are the contact points of the merged circles (blue & tan area), and the triangle of maximum area that fits inside the area between the merged circles (tan area). Note that the "maximum area" triangle is similar to but smaller than the "inner" triangle. Note also that the incircle of the "maximum area" triangle contacts the merged circles at the same point where the "maximum area" triangle contacts the merged circles; this is guaranteed by the fact that the "maximum area" triangle's sides are tangent to the merged circles at the points of greatest intrusion into the space between the merged circles while the "maximum area" incircle is tangent to the sides of the triangle at these same points. Developing a more compact calculation of the area of the "maximum area" triangle requires determining the relationship of the "maximum area" triangle to the similar but larger "inner" triangle.



The distances (d_1, d_2, & d_3) between the sides of the inner triangle and the sides of the maximum area triangle are given by the equations below:

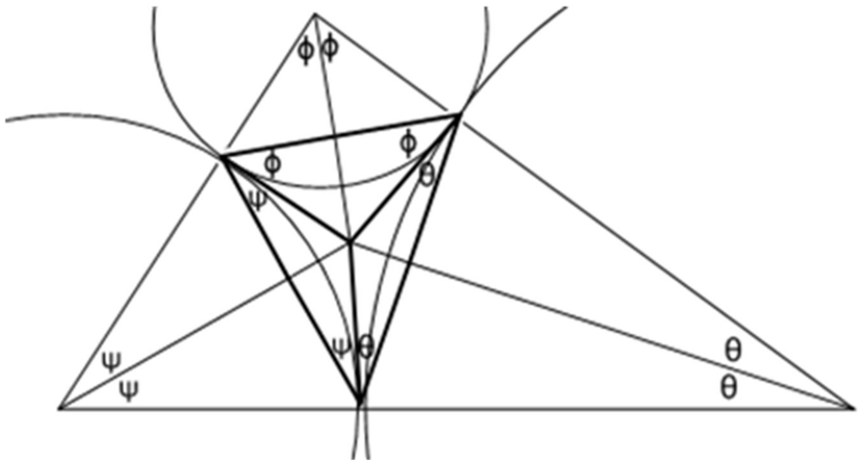


$$d_1 = R_1 - L_1 = R_1 - R_1 \cos\left\{\arctan\left(\frac{r}{R_1}\right)\right\} = R_1 - R_1 \cos(\theta) = R_1\{1 - \cos(\theta)\}$$

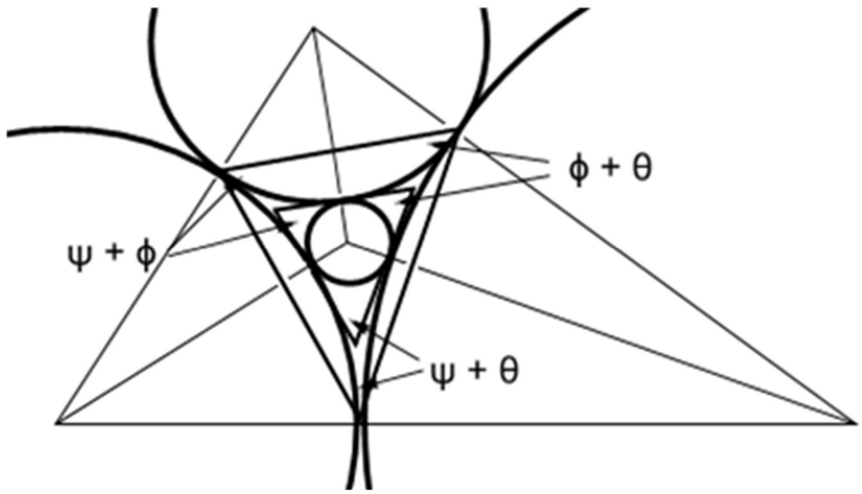
$$d_2 = R_2 - L_2 = R_2 - R_2 \cos\left\{\arctan\left(\frac{r}{R_2}\right)\right\} = R_2 - R_2 \cos(\emptyset) = R_2\{1 - \cos(\emptyset)\}$$

$$d_3 = R_3 - L_3 = R_3 - R_3 \cos\left\{\arctan\left(\frac{r}{R_3}\right)\right\} = R_3 - R_3 \cos(\psi) = R_3\{1 - \cos(\psi)\}$$

Here is a relationship of angles within the inner triangle. The angles inside the inner triangle are $\psi + \theta$, $\psi + \phi$, and $\theta + \phi$.



As the maximum area triangle and the inner triangle are similar, the angles inside the maximum area triangle must also be $\psi + \theta$, $\psi + \phi$, and $\theta + \phi$.



The total area of the space inside the merged circles (A_{insi}) is (angles in degrees):

$$A_{insi} = Area\ A, B, C\ Triangle - \pi\left\{(R_1)^2\left(\frac{2\theta}{360}\right) - (R_2)^2\left(\frac{2\emptyset}{360}\right) - (R_3)^2\left(\frac{2\psi}{360}\right)\right\}$$

$$A = R_1 + R_2 \qquad B = R_1 + R_3 \qquad C = R_2 + R_3 \qquad s = \frac{A + B + C}{2}$$

$$R_1 = \frac{A + B - C}{2} \qquad R_2 = \frac{A + C - B}{2} \qquad R_3 = \frac{B + C - A}{2}$$

$$Area\ A, B, C\ Triangle = \sqrt{s(s - A)(s - B)(s - C)}$$

For the area of the inner triangle (a, b, c) using a, b, and c as calculated above:

$$Area\ a, b, c\ Triangle = \sqrt{s(s - a)(s - b)(s - c)} \qquad s = \frac{a + b + c}{2}$$

By the Descartes Circle Theorem, the radius r of the incircle of the maximum area triangle, note that this circle can also be regarded as the fourth circle within the interior of three kissing circles, is given by:

$$r = \frac{(R_1)(R_2)(R_3)}{(R_1R_2) + (R_1R_3) + (R_2R_3) + 2\sqrt{(R_1)(R_2)(R_3)(R_1 + R_2 + R_3)}}$$

Recall the formula for the incircle (applied here to the maximum area triangle):

$$r = \frac{A}{s} \qquad A = rs \qquad s = \frac{a_{ma} + b_{ma} + c_{ma}}{2}$$

We can check that the area of the maximum area triangle as calculated from the Excel sheet using points and slopes has an incircle radius equal to that indicated by the Descartes Circle Theorem. One can use the (13, 14, 15) scalene triangle ($R_1 = 6$, $R_2 = 7$, $R_3 = 8$). The Excel sheet indicates the following:

Maximum Area Triangle leg lengths = 3.572816361, 3.728228119, 3.840662185
Maximum Area Triangle value of semiperimeter (s) = 5.570853332
Maximum Area Triangle Area (A) = 5.957007816
Maximum Area Triangle calculated Incircle Radius = A/s = 1.069316936

$$r = \frac{(6)(7)(8)}{(6)(7) + (6)(8) + (7)(8) + 2\sqrt{(6)(7)(8)(6 + 7 + 8)}} = 1.070063694$$

Given the propagation of errors in the calculation of this value from points and slopes, the two values are "adequately" equal. We can carry out the same comparison with a (10, 13, 17) scalene triangle ($R_1 = 3$, $R_2 = 7$, $R_3 = 10$). The Excel sheet indicates the following:

Maximum Area Triangle leg lengths = 2.428461722, 3.243895614, 3.400526584
Maximum Area Triangle value of semiperimeter (s) = 4.53644196
Maximum Area Triangle Area (A) = 3.747028517
Maximum Area Triangle calculated Incircle Radius = A/s = 0.825984009

$$r = \frac{(3)(7)(10)}{(3)(7) + (3)(10) + (7)(10) + 2\sqrt{(3)(7)(10)(3 + 7 + 10)}} = 0.837939293$$

.

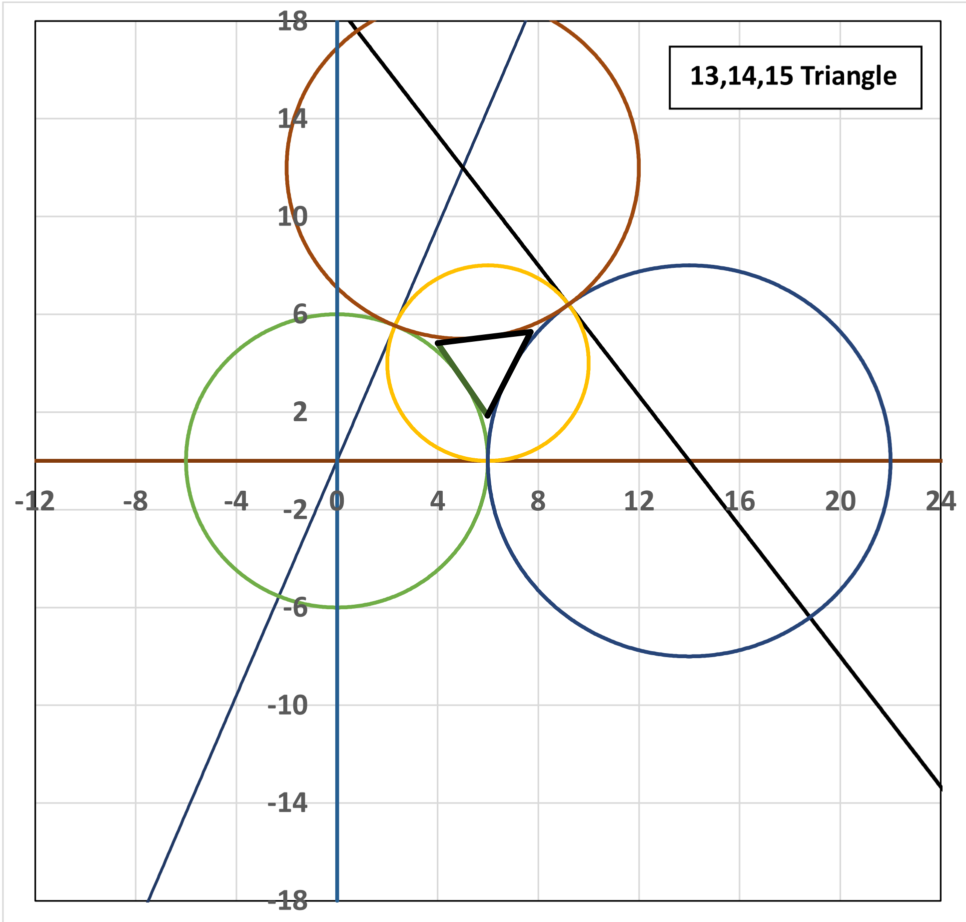
Again, the values are close. We can get better agreement by removing some of the approximation error via use of an equilateral triangle. Here are the values for a (7, 7, 7) equilateral triangle ($R_1 = 3.5$, $R_2 = 3.5$, $R_3 = 3.5$). The Excel sheet indicates the following:

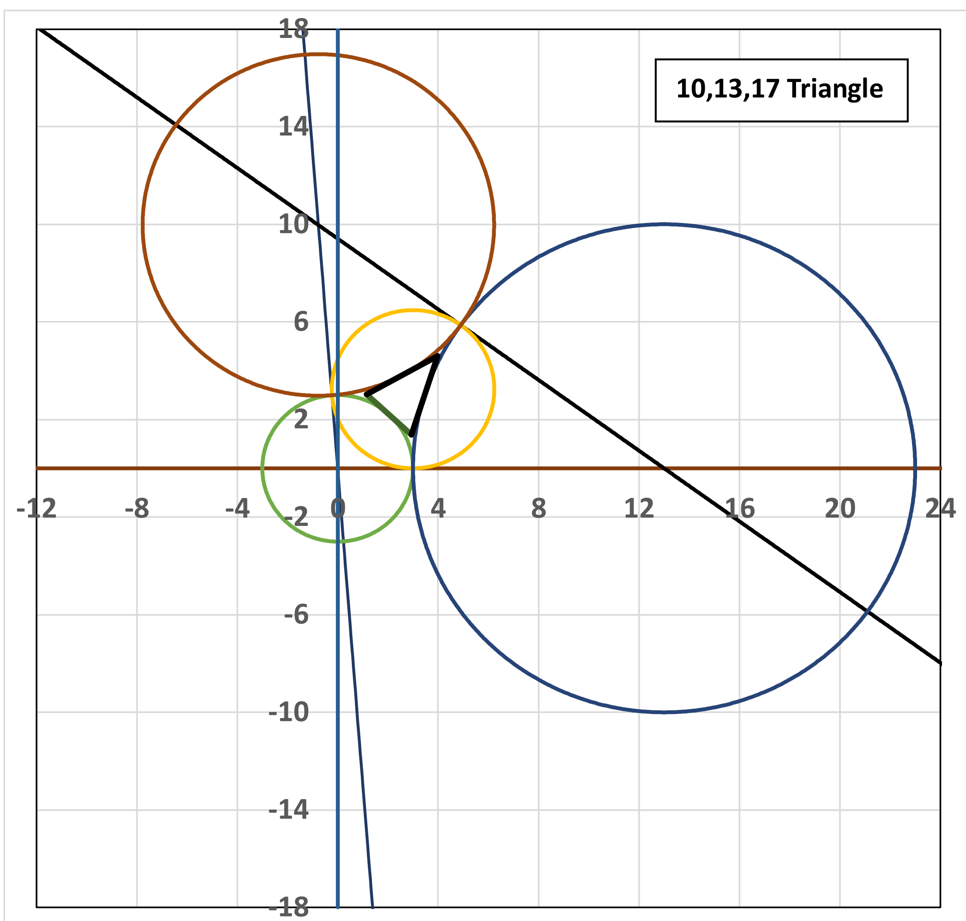
Maximum Area Triangle leg lengths = 1.875644347, 1.875644347, 1.875644347
Maximum Area Triangle value of semiperimeter (s) = 2.813466521
Maximum Area Triangle Area (A) = 1.523356749
Maximum Area Triangle calculated Incircle Radius = A/s = 0.541451884

$$r = \frac{(7)(7)(7)}{(7)(7) + (7)(7) + (7)(7) + 2\sqrt{(7)(7)(7)(7 + 7 + 7)}} = 0.541451884$$

As expected, removal of the rounding errors results in these two values becoming exactly equal. Here is a comparison of these values for a (7, 7 10) triangle: r (Excel sheet) = 0.52600177; r (Descartes) = 0.531972647. The difference in these two values (Excel versus Descartes) increases with the irregularity of the triangle. It is sufficient for an engineer to assume that the Excel method and the Descartes Circle Theorem return the same value.

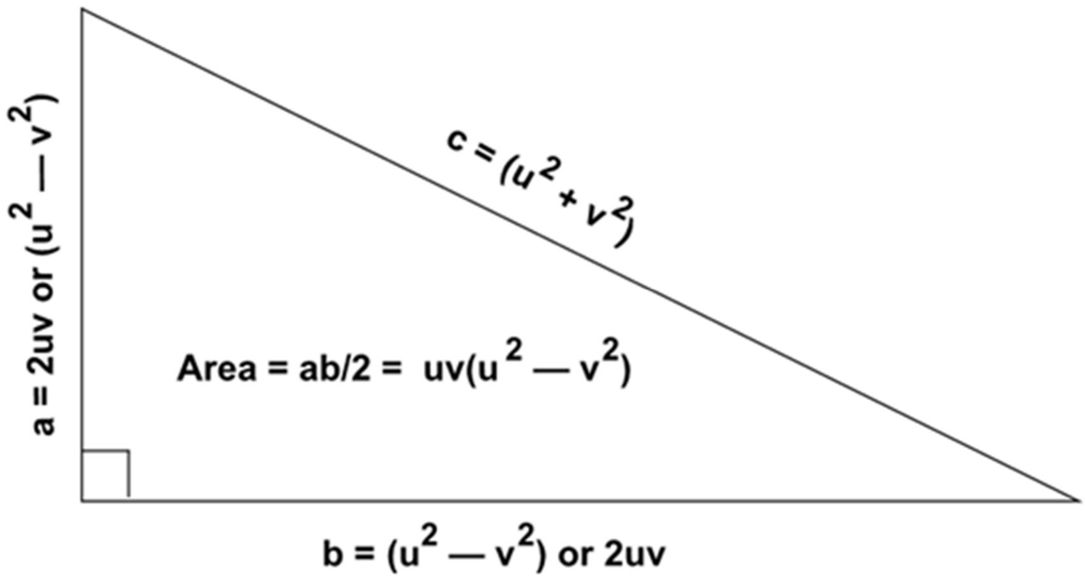
Graphics showing the (13, 14, 15) and (10, 13, 17) maximum area triangles:





Appendix IV: Families of Pythagorean Triangles:

This section (appendix IV) is partially based on the online publication "The Pythagorean Tree: A New Species" by H. Lee Price (Cornell University; online publication arXiv:0809.4324). This publication has also been summarized on YouTube by Burkhard Pollster. The ideas in this section relate to families of right triangles.



Consider the following 2X2 number matrix;

$$\begin{bmatrix} n_1 & n_2 \\ n_4 & n_3 \end{bmatrix} = \begin{bmatrix} u-v & v \\ u+v & u \end{bmatrix}$$

If "u – v" is a positive odd integer and "u – v" & "v" are coprime positive integers, then "u" and "v" fulfill the requirements of the two integers necessary to generate a primitive Pythagorean triple (*i.e.*, one is odd and one is even and they are coprime to each other). If "v" is even, then "u" is odd. If "v" is odd, then "u" is even. If "u – v" and "v" are coprime, then "u" must have no factors in common with "v", and "u" & "v" are thus necessarily coprime. Note also that "u" must necessarily be greater than "v".

If u & v are coprime integers (u > v) with one being even and the other odd, then a primitive Pythagorean triple can be generated as (note that $u^2 - v^2$ & 2uv are the legs of the right triangle corresponding to the triple):

$$u^2 - v^2,\ 2uv,\ u^2 + v^2$$

Observe in the 2X2 matrix above:

$$n_1 \times n_4 = (u - v)(u + v) = u^2 - v^2$$

$$2 \times n_2 \times n_3 = 2uv$$

$$(n_1 \times n_3) + (n_2 \times n_4) = (u - v)(u) + (v)(u + v) = u^2 - vu + vu + v^2 = u^2 + v^2$$

$$\text{Area} = n_1 \times n_2 \times n_3 \times n_4 = (u - v)(v)(u)(u + v) = (uv)(u^2 - v^2) = (\text{height} \times \text{base})/2$$

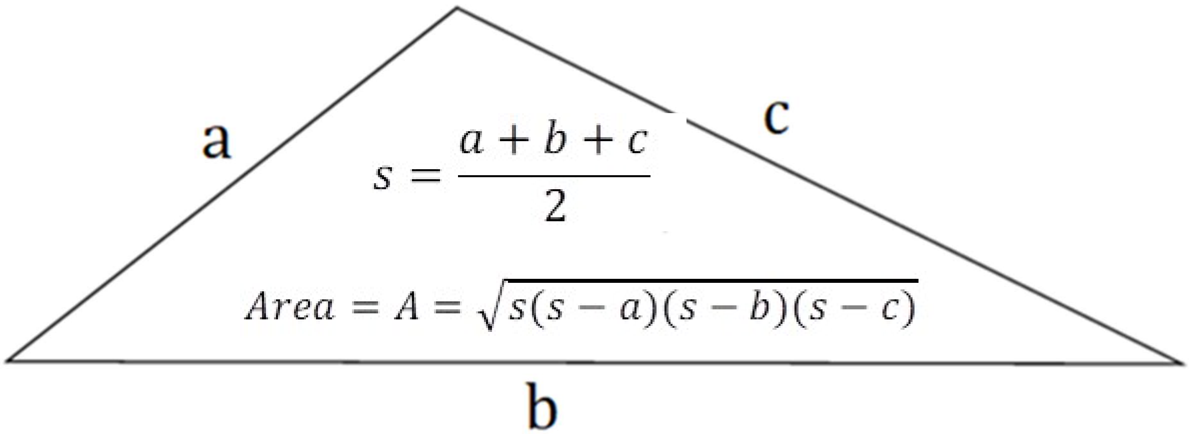
In summary:

$$\begin{bmatrix} n_1 & n_2 \\ n_4 & n_3 \end{bmatrix} = \begin{bmatrix} u-v & v \\ u+v & u \end{bmatrix}$$

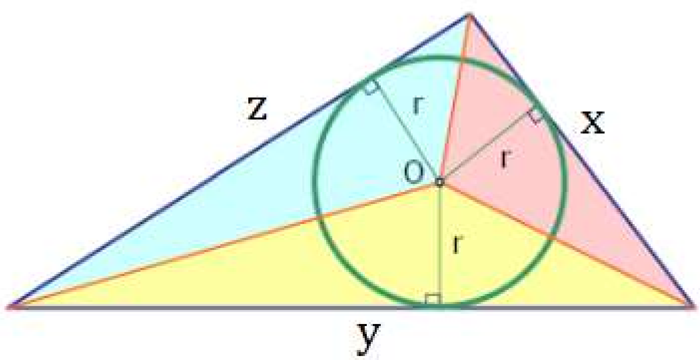
The integer "u – v" is odd and the integers "u – v" & "v" are coprime, then:

Odd leg of right triangle = $n_1n_4 = (u - v)(u + v) = u^2 - v^2$
Even leg of right triangle = $2n_2n_3 = 2(v)(u) = 2uv$
Hypotenuse of right triangle = $n_1n_3 + n_2n_4 = (u - v)(u) + (v)(u + v) = u^2 + v^2$
Area of right triangle = $n_1n_2n_3n_4 = (u - v)(v)(u)(u + v) = (uv)(u^2 - v^2) = ab/2$

Recall Heron's Formula for the area "A" of a triangle. The length of the sides of the triangle are a, b, & c. The semiperimeter "s" of the triangle is defined as the sum of a, b, & c divided by 2. The derivation of Heron's Formula is straightforward and can be found online (Wikipedia, for instance).



A more interesting aspect of the matrix above is how conveniently it allows for calculation of the incircle and excircle radii of a right triangle. See the diagram below for a visualization of the incircle of an arbitrary triangle:



The radius of the incircle is given as "r". Examination of the figure reveals that the radius of the incircle is equal to:

$$Area\ of\ Triangle = A = \frac{(x+y+z)}{2}r = sr \qquad s = \frac{x+y+z}{2} \qquad r = \frac{A}{s}$$

Referring back to Heron's Formula:

$$r = \frac{A}{s} = \frac{\sqrt{s(s-x)(s-y)(s-z)}}{s} = \sqrt{\frac{s(s-x)(s-y)(s-z)}{s^2}}$$
$$= \sqrt{\frac{(s-x)(s-y)(s-z)}{s}}$$

Referring back to the 2 x 2 matrix:

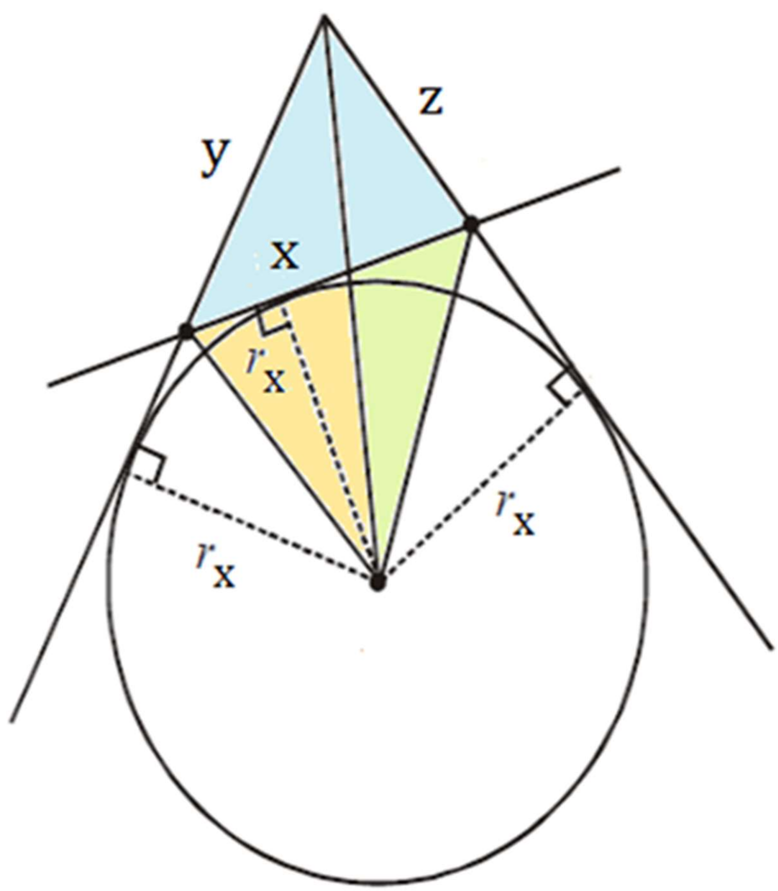
$$\begin{bmatrix} n_1 & n_2 \\ n_4 & n_3 \end{bmatrix} = \begin{bmatrix} u-v & v \\ u+v & u \end{bmatrix}$$

$$r = \frac{A}{s} = \frac{n_1 n_2 n_3 n_4}{\frac{n_1 n_4 + 2n_2 n_3 + n_1 n_3 + n_2 n_4}{2}} = \frac{2n_1 n_2 n_3 n_4}{n_1 n_4 + 2n_2 n_3 + n_1 n_3 + n_2 n_4}$$

$$r = \frac{A}{s} = \frac{2(u-v)(u)(v)(u+v)}{(u-v)(u+v) + 2(uv) + (u-v)(u) + v(u+v)} = \frac{2(uv)(u^2 - v^2)}{2(u^2 + uv)}$$

$$r = \frac{A}{s} = \frac{2(uv)(u^2 - v^2)}{2(u^2 + uv)} = (u-v)(v) = n_1 n_2$$

See the diagram below for visualization of one of the three excircles of a triangle. The excircle shown is the one associated with side "x" of the triangle. Note that r refers to the triangle's incircle radius while r_x refers to the radius of the triangle's x-side excircle:



$$r_x = excircle\ radius \qquad r = incircle\ radius$$

$$Area\ of\ colored\ area = \frac{r_x y}{2} + \frac{r_x z}{2} = Area\ of\ blue\ triangle + \frac{r_x x}{2}$$

$$\frac{r_x(y+z-x)}{2} = Area\ of\ triangle = A = sr \qquad \frac{(y+z-x)}{2} = (s-x)$$

$$r_x(s-x) = A = sr \qquad r_x = \frac{sr}{s-x} = \frac{A}{s-x}$$

$$r_x(s-x) = A = \sqrt{s(s-x)(s-y)(s-z)}$$

$$r_x = \sqrt{\frac{s(s-x)(s-y)(s-z)}{(s-x)^2}} = \sqrt{\frac{s(s-y)(s-z)}{(s-x)}}$$

$$r_x = \frac{x+y+z}{(y+z-x)}r = \frac{(u^2-v^2)+2uv+(u^2+v^2)}{2uv+(u^2+v^2)-(u^2-v^2)}(u-v)(v)$$

$$r_x = \frac{2u^2+2uv}{2v^2+2uv}(u-v)(v) = \frac{u^2+uv}{v^2+uv}(u-v)(v) = \frac{u(u+v)}{v(u+v)}(u-v)(v)$$

$$r_x = \frac{u}{v}(u-v)(v) = (u-v)(u) = n_1n_3$$

$$r_x = n_1n_3$$

$$As\ u^2-v^2\ was\ the\ value\ assigned\ to\ x, the\ x\ side\ here\ was\ the\ n_1n_4\ odd\ leg$$

$$r_x\ corresponds\ to\ the\ excircle\ for\ the\ n_1n_4\ odd\ leg\ of\ the\ right\ triangle$$

Go back to the step where a specific side of the triangle is inserted into the formula and calculate the excircle radii for the even leg of the primitive right triangle:

$$r_y = \frac{x+y+z}{(x+z-y)}r = \frac{(u^2-v^2)+2uv+(u^2+v^2)}{(u^2-v^2)+(u^2+v^2)-2uv}(u-v)(v)$$

$$r_y = \frac{2u^2+2uv}{2u^2-2uv}(u-v)(v) = \frac{u(u+v)}{u(u-v)}(u-v)(v) = (u+v)(v) = n_2n_4$$

$$r_y\ corresponds\ to\ the\ 2n_2n_3\ even\ side\ of\ the\ right\ triangle$$

Finally, calculate the radius for the excircle of the hypotenuse:

$$r_z = \frac{2(area\ of\ triangle)}{(x+y-z)} = \frac{2(u-v)(v)(u)(u+v)}{(u^2-v^2)+2uv-(u^2+v^2)}$$

$$r_z = \frac{2(u^2-v^2)(v)(u)}{2uv-2v^2} = \frac{(u^2-v^2)(u)}{(u-v)} = \frac{(u-v)(u+v)u}{(u-v)} = u(u+v)$$

$$r_z = (u)(u+v)$$

$$(u)(u+v) = n_3 n_4$$

$$r_z \ corresponds \ to \ the \ n_1 n_3 + n_2 n_4 \ hypotenuse \ of \ the \ right \ triangle$$

$$r_z = \frac{x+y+z}{(x+y-z)} r = \frac{(u^2-v^2)+2uv+(u^2+v^2)}{(u^2-v^2)+2uv-(u^2+v^2)}(u-v)(v)$$

$$r_z = \frac{2u^2+2uv}{2uv-2v^2}(u-v)(v) = \frac{u(u+v)}{v(u-v)}(u-v)(v) = (u)(u+v) = n_3 n_4$$

<u>Summary:</u>

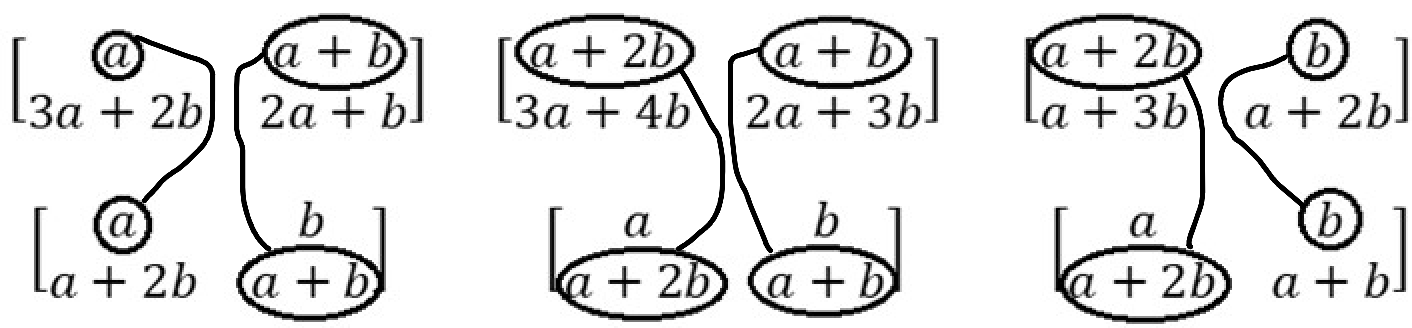
Choose "u – v" to be an odd integer and choose "v" coprime to "u – v".

$$\begin{bmatrix} n_1 & n_2 \\ n_4 & n_3 \end{bmatrix} = \begin{bmatrix} (u-v) & v \\ (u+v) & u \end{bmatrix}$$

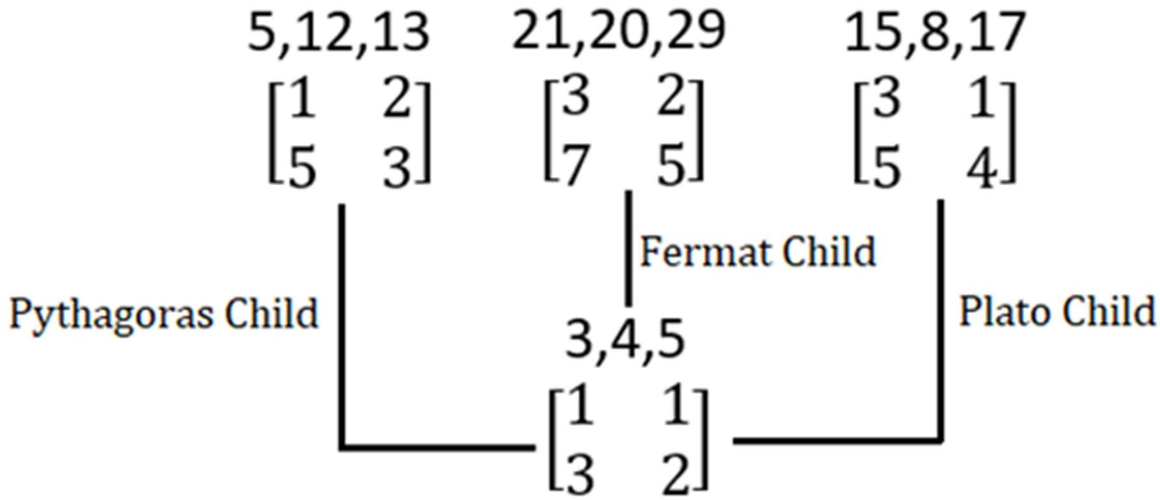
$$length \ of \ odd \ leg \ of \ right \ triangle = n_1 n_4 = u^2 - v^2$$
$$length \ of \ even \ leg \ of \ right \ triangle = 2n_2 n_3 = 2uv$$
$$length \ of \ hypotenuse \ of \ right \ triangle = n_1 n_3 + n_2 n_4 = u^2 + v^2$$
$$Area \ of \ right \ triangle = n_1 n_2 n_3 n_4 = uv(u^2 - v^2)$$
$$incircle \ radius = n_1 n_2 = (u-v)(v)$$
$$excircle \ radius \ of \ odd \ leg = n_1 n_3 = (u-v)(u)$$
$$excircle \ radius \ of \ even \ leg = n_2 n_4 = (v)(u+v)$$
$$excircle \ radius \ of \ hypotenuse = n_3 n_4 = (u)(u+v)$$

<u>Families of Pythagorean Triples:</u>

Note that the excircle of a given right triangle can serve as the incircle of a larger right triangle:



If "a" is odd and "a" & "b" are coprime, then each of the "children" of the "parent" right triangle have an odd number in position n_1 and the numbers in position n_1 & n_2 are coprime. If the "parent" triangle is a primitive right triangle, then the "child" triangles are also primitive right triangles. The example below shows the "children" of the primitive 3,4,5 right triangle.



Note that for the "Plato child", the hypotenuse and the larger leg differ by two. For the "Fermat child", the two legs differ by one. For the "Pythagoras child", the hypotenuse and the larger leg differ by one. These relations will continue so long as one continues to take the same type of "child". It is possible to work back from an arbitrary primitive right triangle to the base 3,4,5 triangle (see graphic below). The triangle on the top is the larger "child" triangle. The triangles below are the possible "parent" triangles.

$$\begin{bmatrix}a & b\\a+2b & a+b\end{bmatrix}$$

$$\begin{bmatrix}a & b-a\\2b-a & b\end{bmatrix} \quad \begin{bmatrix}2b-a & a-b\\a & b\end{bmatrix} \quad \begin{bmatrix}a-2b & b\\a & a-b\end{bmatrix}$$

$$b > a \qquad\qquad 2b > a > b \qquad\qquad a > 2b$$

Note that "a" can't equal "2b", as "a" is odd ("b" can equal "2a"). Also, "a" can't equal "b", as this would require a matrix entry of zero. One simply looks at the "a" and "b" integers in the matrix and determines what type of child it is. Next apply the appropriate matrix to find the parent. See the following example:

$$20907, 16324, 26525$$

$$\begin{bmatrix}101 & 53\\207 & 154\end{bmatrix}$$

$$2b > a > b \qquad \text{"}Fermat\ Child\text{"}$$

$$505, 5088, 5113$$

$$\begin{bmatrix}5 & 48\\101 & 53\end{bmatrix}$$

$$b > a \qquad \text{"}Pythagoras\ Child\text{"}$$

$$455, 4128, 4153$$

$$\begin{bmatrix} 5 & 43 \\ 91 & 48 \end{bmatrix}$$

$b > a$ "*Pythagoras Child*"

$$405, 3268, 3293$$

$$\begin{bmatrix} 5 & 38 \\ 81 & 43 \end{bmatrix}$$

$b > a$ "*Pythagoras Child*"

$$355, 2508, 2533$$

$$\begin{bmatrix} 5 & 33 \\ 71 & 38 \end{bmatrix}$$

$b > a$ "*Pythagoras Child*"

$$305, 1848, 1873$$

$$\begin{bmatrix} 5 & 28 \\ 61 & 33 \end{bmatrix}$$

$b > a$ "*Pythagoras Child*"

$$255, 1288, 1313$$

$$\begin{bmatrix} 5 & 23 \\ 51 & 28 \end{bmatrix}$$

$b > a$ "*Pythagoras Child*"

$$205, 828, 853$$

$$\begin{bmatrix} 5 & 18 \\ 41 & 23 \end{bmatrix}$$

$b > a$ "*Pythagoras Child*"

$$155, 468, 493$$

$$\begin{bmatrix} 5 & 13 \\ 31 & 18 \end{bmatrix}$$

$b > a$ $\quad$ $"Pythagoras\ Child"$

$$105, 208, 233$$

$$\begin{bmatrix} 5 & 8 \\ 21 & 13 \end{bmatrix}$$

$b > a$ $\quad$ $"Pythagoras\ Child"$

$$55, 48, 73$$

$$\begin{bmatrix} 5 & 3 \\ 11 & 8 \end{bmatrix}$$

$2b > a > b$ $\quad$ $"Fermat\ Child"$

$$5, 12, 13$$

$$\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$$

$b > a$ $\quad$ $"Pythagoras\ Child"$

$$3, 4, 5$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$$

$"Base\ Right\ Triangle"$

There can't be a smaller integer right triangle than the "base right triangle", as the incircle radius of the "base right triangle" is one and there is no positive integer less than one. For the "base right triangle" to have a "parent", one of the excircle radii of this hypothetical parent triangle would necessarily be equal to one, and one of the two following matrices is therefore necessary.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \textcircled{1} & 0 \\ 1 & \textcircled{1} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ \textcircled{1} & \textcircled{1} \end{bmatrix} \quad \begin{bmatrix} -1 & \textcircled{1} \\ \textcircled{1} & 0 \end{bmatrix}$$

Both of these possible parent matrices contain a zero, and thus one leg of these hypothetical triangles will be of zero length. Neither of these matrices form a triangle. Thus, no integer right triangle smaller than the 3,4,5 "base right triangle" is possible.

<u>Proof all primitive Pythagorean Triples are generated as "children" of the base 3,4,5 triangle:</u>

Any primitive right triangle (*i.e.*, primitive Pythagorean triple) can be represented by a 2X2 matrix. Choose two coprime integers u & v with one being even and one odd and with u > v. The integer "u – v" is odd. The odd leg of the Pythagorean triple is equal to $u^2 - v^2$. The even leg of the Pythagorean triple is equal to 2uv. The hypotenuse of the Pythagorean triple is $u^2 + v^2$. All integer primitive Pythagorean triples can be created this way. Take any integer primitive Pythagorean triple represented as a 2 x 2 matrix:

$$\begin{bmatrix} n_1 & n_2 \\ n_4 & n_3 \end{bmatrix} = \begin{bmatrix} u-v & v \\ u+v & u \end{bmatrix}$$

$$odd\ leg\ of\ right\ triangle = n_1 n_4 = u^2 - v^2$$
$$even\ leg\ of\ right\ triangle = 2n_2 n_3 = 2uv$$
$$hypotenuse\ of\ right\ triangle = n_1 n_3 - n_2 n_4 = u^2 + v^2$$
$$incircle\ radius\ of\ right\ triangle = n_1 n_2$$
$$area\ of\ right\ triangle = n_1 n_2 n_3 n_4 = uv(u-v)(u+v)$$
$$excircle\ radius\ of\ odd\ leg\ of\ right\ triangle = n_1 n_3$$
$$excircle\ radius\ of\ even\ leg\ of\ right\ triangle = n_2 n_4$$
$$excircle\ radius\ of\ hypotenuse\ of\ right\ triangle = n_3 n_4$$

Such a matrix can be converted to a smaller parent matrix which can in turn be converted to a still smaller parent matrix until it reaches the base 3,4,5 right triangle. Any primitive Pythagorean triple larger than the base 3,4,5 triple can be reduced to a smaller Pythagorean triple by converting it to its parent. By infinite descent; any integer primitive Pythagorean triple with an incircle radius greater than one can be reduced to a smaller integer parent triple and this process can be repeated until it reaches the base 3,4,5 triple, and no integer Pythagorean triple smaller than the base 3,4,5 triple is possible.

Assume that the "child" triple is primitive. There are three possible parents:

$$\begin{bmatrix} u-v & v \\ u+v & u \end{bmatrix}$$

$$\begin{bmatrix} u-3v & v \\ u-v & u-2v \end{bmatrix} \quad \begin{bmatrix} 3v-u & u-2v \\ u-v & v \end{bmatrix} \quad \begin{bmatrix} u-v & 2v-u \\ 3v-u & v \end{bmatrix}$$

$$u - v > 2v \qquad 2v > u - v > v \qquad v > u - v$$

If “u” & “v” are coprime, then “v”, “u – 2v”, & “2v – u” are coprime. If one of “u” & “v” is odd and the other even (*i.e.*, “u” and “v” have opposite parity), then “v” and “u – 2v” have opposite parity and also “v” and “2v – u” have opposite parity. Thus, the parent of a primitive Pythagorean triple is also a primitive Pythagorean triple.

Rational Approximation of the Square Root of 2:

Consider the Fermat leg of the Pythagorean triples generated with a 2X2 matrix. Here are the first five triples generated:

$$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\boxed{3,4,5}$$

$$\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

$$\boxed{21,20,29}$$

$$\begin{bmatrix} 7 & 5 \\ 17 & 12 \end{bmatrix}$$

$$\boxed{119,120,169}$$

$$\begin{bmatrix} 17 & 12 \\ 41 & 29 \end{bmatrix}$$

$$\boxed{697,696,985}$$

$$\begin{bmatrix} 41 & 29 \\ 99 & 70 \end{bmatrix}$$

$$\boxed{4059,4060,5741}$$

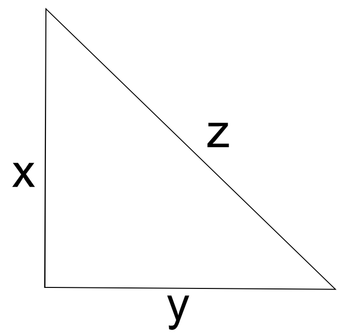
$$\begin{bmatrix} 99 & 70 \\ 239 & 169 \end{bmatrix}$$

$$\boxed{23661,23660,33461}$$

$$\begin{bmatrix} 239 & 169 \\ 577 & 408 \end{bmatrix}$$

$$\boxed{137903,137904,195025}$$

Note that the Pythagorean triples generated in the Fermat leg of this method have two legs that differ in magnitude by one. As these Pythagorean triples increase in magnitude, the two legs approach equal length. In the infinite limit, these two legs are equal in length. Thus:



$$x^2 + y^2 = z^2$$

$$z = \sqrt{x^2 + y^2}$$

$$\lim_{x,y\to\infty} \sqrt{x^2 + y^2} = \sqrt{2}\sqrt{x^2} = \sqrt{2}\sqrt{y^2}$$

$$\lim_{x,y\to\infty} \left(\frac{z}{x}\right) = \lim_{x,y\to\infty} \left(\frac{z}{y}\right) = \sqrt{2}$$

Thus, the value of the hypotenuse divided by either leg from a Pythagorean triple in the Fermat leg is an approximation of the square root of two. Using the first seven triples:

Leg 1	Leg 2	Hypotenuse	(Hypotenuse)/(Leg 1)	% Error	(Hypotenuse)/(Leg 2)	% Error
3	4	5	1.666666667	17.8511	1.25	11.6117
20	21	29	1.45	2.5305	1.380952381	2.3519
119	120	169	1.420168067	0.4210	1.408333333	0.4158
696	697	985	1.415229885	0.0719	1.413199426	0.0717
4059	4060	5741	1.41438778	0.0123	1.414039409	0.0123
23660	23661	33461	1.414243449	0.0021	1.414183678	0.0021
137903	137904	195025	1.41421869	0.0004	1.414208435	0.0004

We can compare this approximation to one obtained with an infinite continued fraction:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}}$$

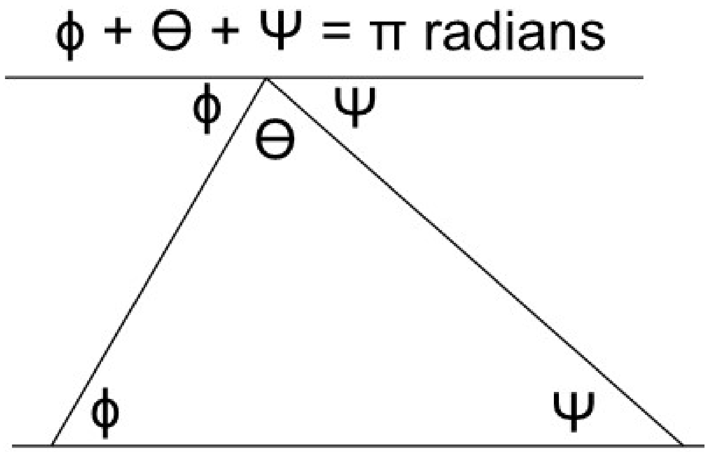
The Table below illustrates a convenient way to generate partial quotients for a continued fraction. Here are the first seven partial quotients:

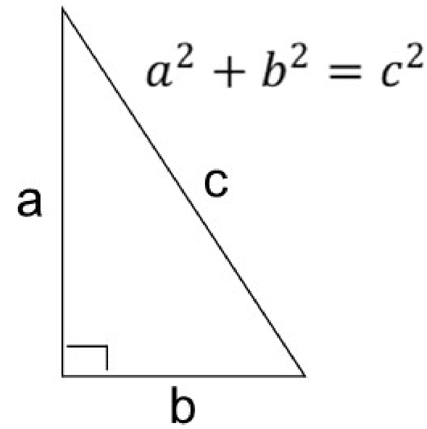
		1	2	2	2	2	2	2
0	1	1	3	7	17	41	99	239
1	0	1	2	5	12	29	70	169
Partial Quotient		1/1	3/2	7/5	17/12	41/29	99/70	239/169
Decimal Quotient		1	1.5	1.4	1.416666667	1.413793103	1.414286	1.414201183
% Error		29.2893	6.0660	1.0051	0.1735	0.0297	0.0051	0.0009

The value obtained with the Fermat leg of the Pythagorean triples converges a bit faster per step, but the continued fraction converges with significantly smaller fractions per step. Also, the continued fraction method involves significantly fewer calculations per step. The continued fraction method is better.

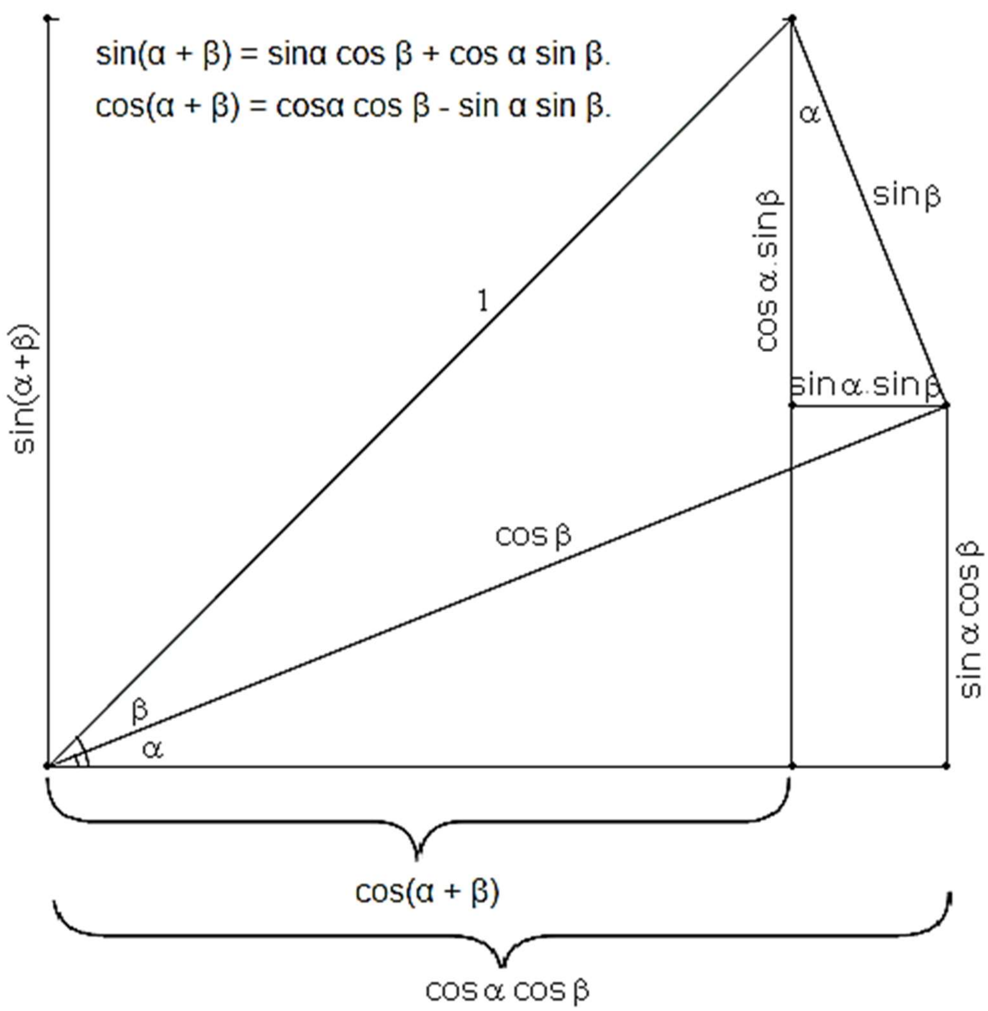
Appendix V: Derivation of the QM ≥ AM ≥ GM ≥ HM relationship from Axioms:

AXIOM 1 & AXIOM 2: Parallel Line Postulate of Euclid; the sum of the three internal angles of a triangle is equal to π radians (180 degrees). This is an alternative statement of Euclid's parallel line postulate. Note that the Pythagorean Theorem extends to an arbitrary number of dimensions with many proofs available.

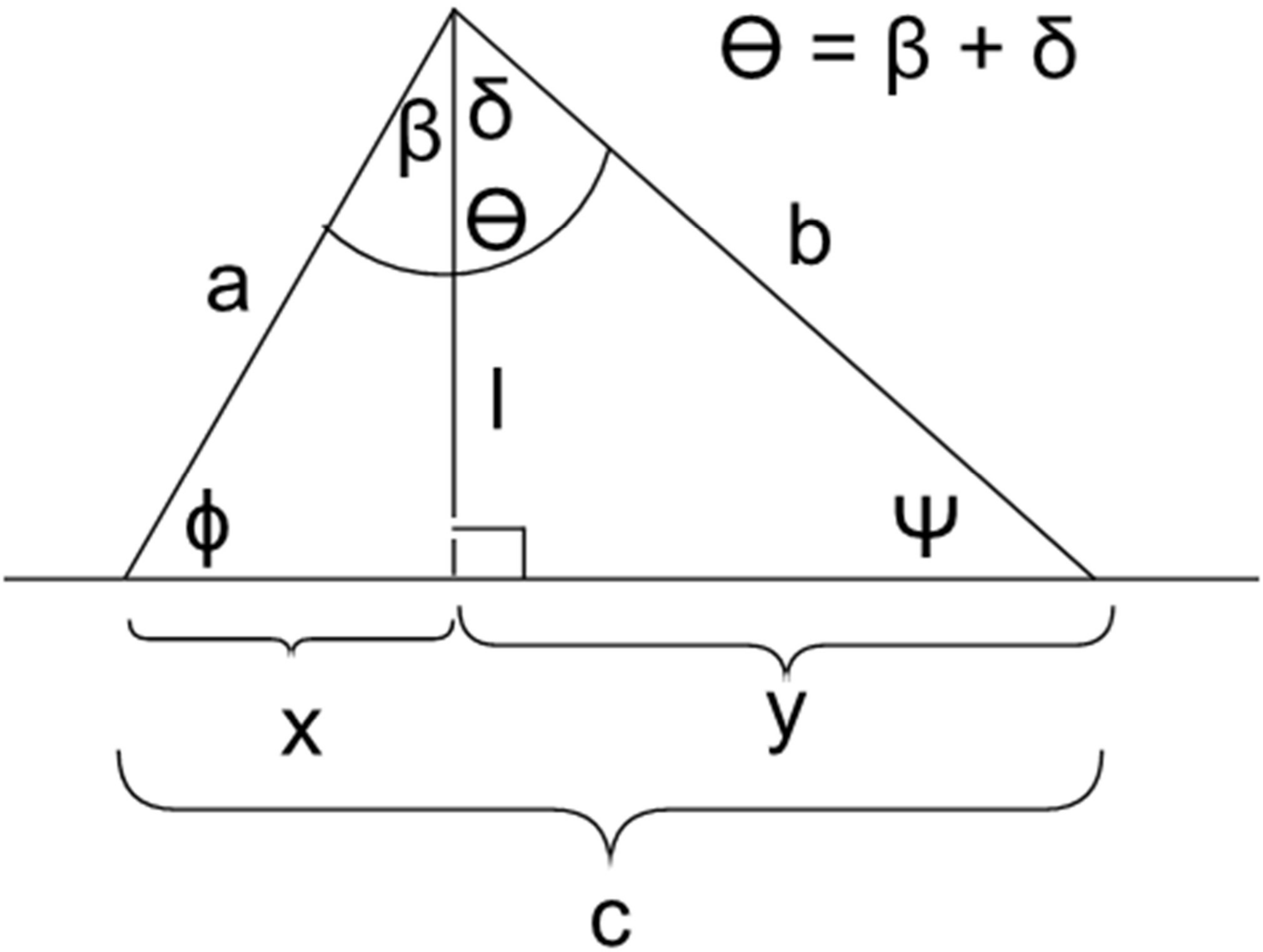




Derivation of cosine of sum of angles by AXIOM 1 (sum internal ∠'s of Δ = 180 degrees):



Derivation of the Law of Cosines from the Pythagorean Theorem:



$$x^2 + l^2 = a^2 \qquad y^2 + l^2 = b^2$$

$$a^2 + b^2 = 2l^2 + x^2 + y^2$$

$$c^2 = (x + y)^2 = x^2 + y^2 + 2xy$$

$$c^2 - 2xy = x^2 + y^2$$

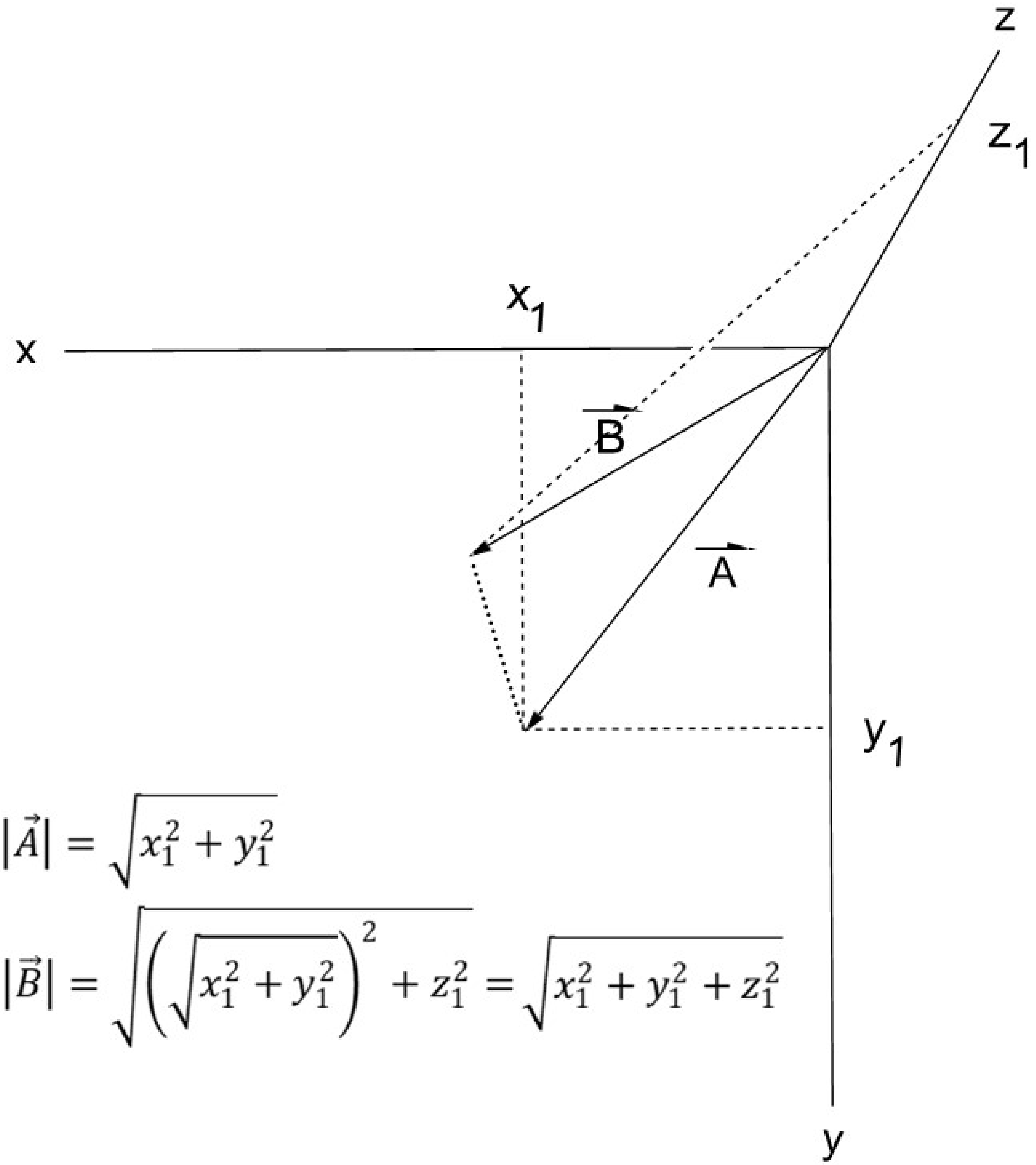
$$a^2 + b^2 = c^2 + 2l^2 - 2xy$$

$$l = acos(\beta) = bcos(\delta) \qquad x = \ asin(\beta) = bsin(\delta)$$

$$a^2 + b^2 = c^2 + 2abcos(\beta)\cos{(\delta)} - 2absin(\beta)\sin{(\delta)}$$

$$a^2 + b^2 = c^2 + 2abcos(\beta + \delta) = c^2 + 2abcos(\theta)$$

Derivation of the length of an "n" dimensional vector:



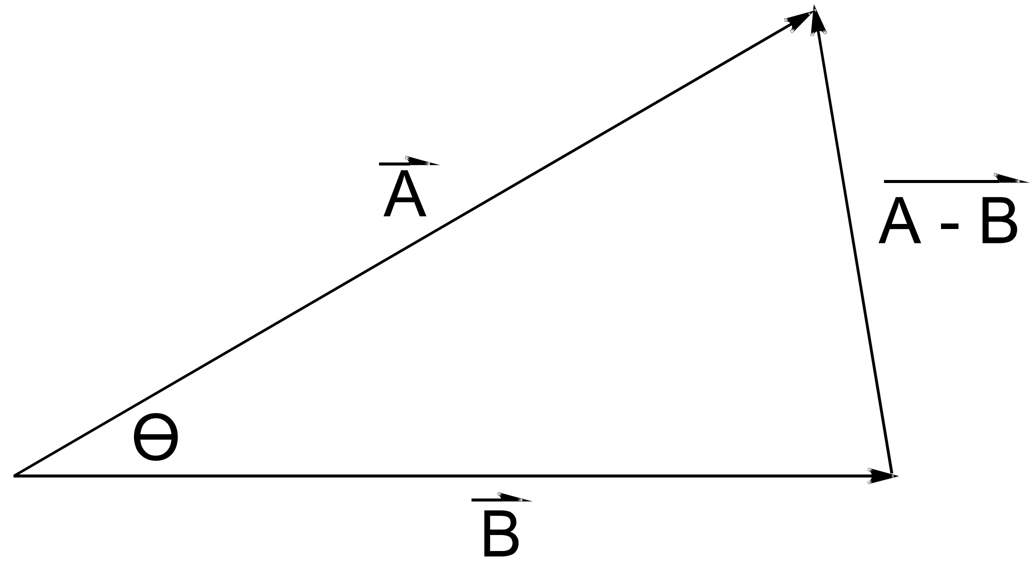
$$|\vec{A}| = \sqrt{x_1^2 + y_1^2}$$

$$|\vec{B}| = \sqrt{\left(\sqrt{x_1^2 + y_1^2}\right)^2 + z_1^2} = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

The length of a vector in 2D (*e.g.*, vector A) is given by the Pythagorean Theorem. If one extends a 2D vector into 3D space (*e.g.*, vector B), one gets the new length by another application of the Pythagorean Theorem. Now imagine the 3D vector moved into a 2D space obtained by rotating the coordinate system and again extending this formerly 3D vector into a new dimension "d" (effectively a 4D vector). The new length is obtained by another application of the Pythagorean Theorem, and the process is continued on to "n" dimensions.

$$|\vec{V}| = \sqrt{x_1^2 + y_1^2 + z_1^2 + d_1^2 + \cdots + n_1^2}$$

Derivation of the Cauchy-Schwarz Inequality via Pythagorean Theorem & The Law of Cosines:



Consider n-dimensional vectors:

$$\vec{A} = x_a + y_a + z_a + \cdots + n_a$$

$$\vec{B} = x_b + y_b + z_b + \cdots + n_b$$

$$\overrightarrow{A - B} = (x_a - x_b) + (y_a - y_b) + (z_a - z_b) + \cdots + (n_a - n_b)$$

$$|\bar{A}| = \sqrt{x_a^2 + y_a^2 + z_a^2 + \cdots + n_a^2}$$

$$|\bar{B}| = \sqrt{x_b^2 + y_b^2 + z_b^2 + \cdots + n_b^2}$$

$$|\overline{A - B}| = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2 + \cdots + (n_a - n_b)^2}$$

Apply The Law of Cosines:

$$|\bar{A}|^2 + |\bar{B}|^2 = |\overline{A - B}|^2 + 2|\bar{A}||\bar{B}|\cos(\theta)$$

$$2x_a x_b + 2y_a y_b + 2z_a z_b + \cdots + 2n_a n_b = 2|\bar{A}||\bar{B}|\cos(\theta)$$

$$\vec{A} \cdot \vec{B} = x_a x_b + y_a y_b + z_a z_b + \cdots + n_a n_b = |\bar{A}||\bar{B}|\cos(\theta)$$

$$|\bar{A}||\bar{B}| \geq \vec{A} \cdot \vec{B} = x_a x_b + y_a y_b + z_a z_b + \cdots + n_a n_b$$

Proof that Quadratic Mean ≥ Arithmetic Mean via The Cauchy-Schwarz Inequality:

$$data = \{x_1, x_2, x_3, \dots, x_n\} = \text{"n"}\ dimensional\ vector$$

$$Quadtric\ Mean = QM = \frac{\sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}}{n}$$

$$Arithmetic\ Mean = AM = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

Consider the dot product of an “n” dimensional vector of all ones with our data vector:

$$\{x_1, x_2, x_3, \dots, x_n\} \cdot \{1, 1, 1, \dots, 1\} = x_1 + x_2 + x_3 + \cdots + x_n$$

$$|\{1, 1, 1, \dots, 1\}| = \sqrt{n}$$

$$|\{x_1, x_2, x_3, \dots, x_n\}| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

$$\sqrt{n}\sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2} \geq x_1 + x_2 + x_3 + \cdots + x_n$$

$$n(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2) \geq (x_1 + x_2 + x_3 + \cdots + x_n)^2$$

$$\frac{n(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)}{n^2} \geq \frac{(x_1 + x_2 + x_3 + \cdots + x_n)^2}{n^2}$$

$$\sqrt{\frac{(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)}{n}} \geq \sqrt{\frac{(x_1 + x_2 + x_3 + \cdots + x_n)^2}{n^2}}$$

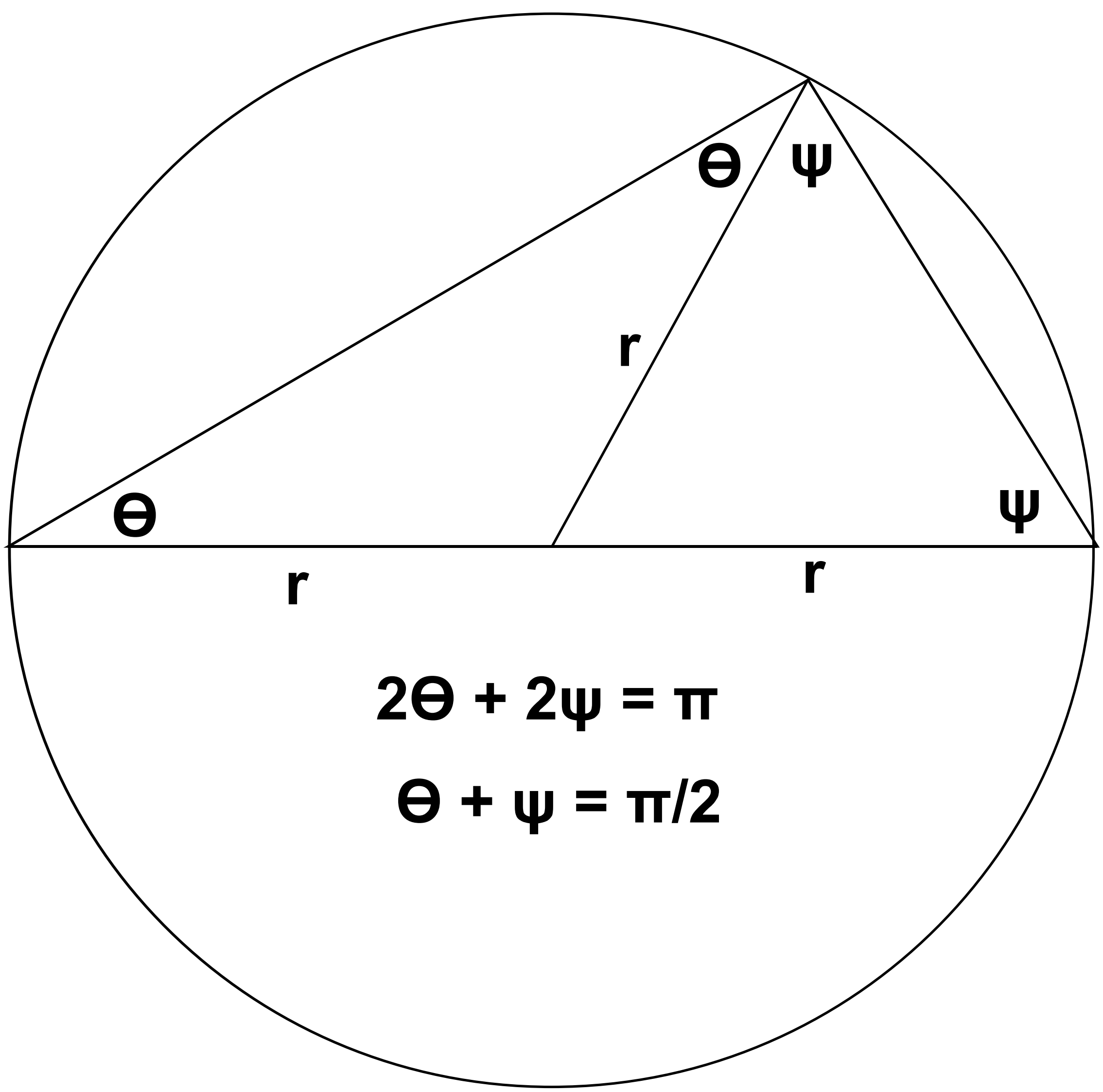
$$\sqrt{\frac{(x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2)}{n}} \geq \frac{(x_1 + x_2 + x_3 + \cdots + x_n)}{n}$$

$$QM = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}} \geq \frac{\sum_{i=1}^{n} x_i}{n} = AM$$

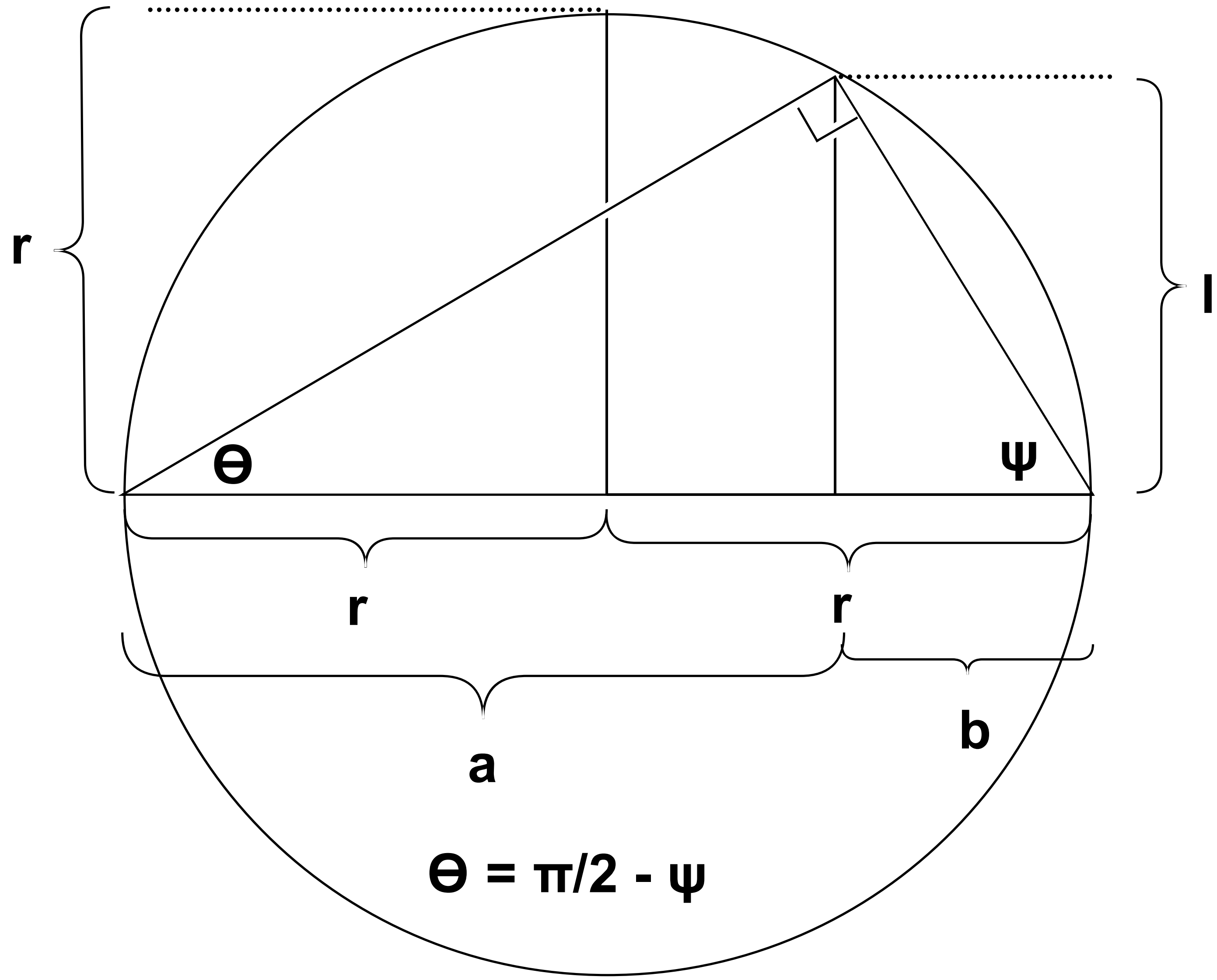
$$QM = AM \; only \; when \; all \; values \; of \; x_i \; are \; identical$$

Limited Proof that the Arithmetic Mean is greater than or equal to the Geometric Mean:

Limited form of the Thales Central Angle Theorem:



Limited proof that AM ≥ GM:



$$AM = \frac{a+b}{2} = r \qquad GM = \sqrt{ab}$$

$$\frac{l}{a} = tan(\theta) \qquad \frac{l}{b} = tan(\psi) \qquad \theta + \psi = \frac{\pi}{2} \qquad tan(\theta) = \cot(\psi)$$

$$\frac{l}{a} = \frac{b}{l} \qquad l^2 = ab \qquad l = \sqrt{ab} = GM$$

$$As\ l \leq r\ with\ equality\ only\ when\ a = b = r \qquad AM \geq GM$$

Full Proof that the Arithmetic Mean ≥ Geometric Mean:

$$Arithmetic\ Mean = AM = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \alpha$$

$$Geometric\ Mean = GM = (x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}}$$

With "α" being equal to the arithmetic mean, pick two values of x_i such that one is greater than α and one is less than α. Assume that x_1 and x_2 meet this criterion with x_2 being great than α and x_1 being less than α, as these can be assigned arbitrarily. If there is no value of x_i greater than α with at least one different value of x_i obligatorily less than α, then all the values of x_i are equal to α and "AM = GM". Substitute the values of "α" & "$x_1 + x_2 - \alpha$" into both the AM and GM:

$$AM = \frac{\alpha + (x_1 + x_2 - \alpha) + x_3 + \cdots + x_n}{n} = \alpha$$

$$GM = \{\alpha(x_2 + x_1 - \alpha)x_3 \cdots x_n\}^{\frac{1}{n}} = \{(\alpha x_2 + \alpha x_1 - \alpha^2)x_3 \cdots x_n\}^{\frac{1}{n}}$$

$$\alpha x_2 + \alpha x_1 - \alpha^2 - x_1 x_2 = (x_2 - \alpha)(\alpha - x_1)$$

$$x_2 - \alpha > 0 \quad \& \quad \alpha - x_1 > 0 \quad \& \quad \alpha x_2 + \alpha x_1 - \alpha^2 > x_1 x_2$$

The geometric mean has been increased by this substitution while the arithmetic mean has remained the same. Now reassign values:

$$Let\ x_1 + x_2 - \alpha = new\ value\ of\ x_1$$

$$Reassign\ x_3\ through\ x_n\ as\ x_2\ through\ x_{n-1}$$

$$New\ Arithmetic\ Mean = \frac{\alpha + x_1 + x_2 + x_3 + \cdots + x_{n-1}}{n} = \alpha$$

$$\alpha + x_1 + x_2 + x_3 + \cdots + x_{n-1} = n\alpha - \alpha = \alpha(n-1)$$

$$New\ Arithmetic\ Mean = \frac{x_1 + x_2 + x_3 + \cdots + x_{n-1}}{n-1} = \alpha$$

So now one can repeat the process. There must be at least one value of x_i greater than α and one value of x_i less than α unless all the values of x_1 through x_{n-1} are equal to α. We arbitrarily assign the value of x_2 to be greater than α and x_1 to be less than α. One continues the process over and over until all the values of x_i have been converted to α. Note that the final two values, assuming they are different from α, will both be converted to α.

$$New\ Arithmetic\ Mean = \frac{x_1 + x_2}{2} = \frac{(\alpha + z\alpha) + (\alpha - z\alpha)}{2} = \alpha$$

$$x_1 + x_2 - \alpha = \alpha + z\alpha + \alpha - z\alpha - \alpha = \alpha$$

With each step, the geometric mean increases while the arithmetic mean remains the same. Ultimately, the value of the geometric mean reaches α.

$$Increased\ Geometric\ Mean = (\alpha\alpha\alpha\alpha\alpha\alpha \cdots \alpha)^{\frac{1}{n}} = (\alpha^n)^{\frac{1}{n}} = \alpha$$

$$Original\ Arithmetic\ Mean = \alpha$$

The original geometric mean was thus less than or equal to the original arithmetic mean.

<u>Proof that the Geometric Mean ≥ Harmonic Mean:</u>

$$Geometric\ Mean = GM = (x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}}$$

$$Harmonic\ Mean = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}$$

Note that the harmonic mean is the inverse of the arithmetic mean of the inverses. The inverse of the geometric mean of the inverses is given below:

$$\frac{1}{Geometric\ Mean\ of\ inverses} = \frac{1}{\left(\left(\frac{1}{x_1}\right)\left(\frac{1}{x_2}\right)\left(\frac{1}{x_3}\right)\cdots\left(\frac{1}{x_n}\right)\right)^{\frac{1}{n}}}$$

$$\frac{1}{Geometric\ Mean\ of\ inverses} = \left(\frac{1}{\left(\left(\frac{1}{x_1}\right)\left(\frac{1}{x_2}\right)\left(\frac{1}{x_3}\right)\cdots\left(\frac{1}{x_n}\right)\right)}\right)^{\frac{1}{n}}$$

$$\frac{1}{Geometric\ Mean\ of\ inverses} = (x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}} = GM$$

By previous proofs:

$$\frac{1}{AM\ of\ inverses} \le \frac{1}{GM\ of\ inverses} = (x_1 x_2 x_3 \cdots x_n)^{\frac{1}{n}} = GM$$

$$\frac{1}{AM\ of\ inverses} = HM \le GM$$

<u>The Weighted Average (Mean):</u>

A weighted average (mean) is convenient when handling data for large populations. For instance, an average test score for a test graded on a scale of zero to one hundred (integers only) for 10,000 students will obviously contain many instances of students with the same grade (pigeon hole principle). Going forward, let "m" designate the value being averaged; in this case, the test grade. Rather than listing all the identical grades individually, one can use a weighted average.

$$n_i = number\ of\ data\ points\ with\ value\ m_i$$

$$\sum_{i=1}^{n} n_i = N = total\ number\ of\ data\ points$$

$$Weighted\ Arithmetic\ Mean = M_n = \frac{\sum_{i=1}^{n} n_i m_i}{\sum_{i=1}^{n} n_i} = \frac{\sum_{i=1}^{n} n_i m_i}{N}$$

The weighted average can be restated with fractions:

$$M_n = \frac{\sum_{i=1}^{n} n_i m_i}{N} = \sum_{i=1}^{n} x_i m_i \qquad x_i = \frac{n_i}{N}$$

Here is the weighted quadratic mean:

$$Weighted\ Quadratic\ Mean = Q_n = \sqrt{\frac{\sum_{i=1}^{n} n_i m_i^2}{\sum_{i=1}^{n} n_i}} = \sqrt{\frac{\sum_{i=1}^{n} n_i m_i^2}{N}}$$

$$Q_n = \sqrt{\frac{\sum_{i=1}^{n} n_i m_i^2}{N}} = \sum_{i=1}^{n} \sqrt{x_i m_i^2}$$

Here is the weighted geometric mean (does not simplify as much as the others):

$$G_n = \left\{\prod_{i=1}^{n} n_i m_i\right\}^{\frac{1}{n}}$$

Here is the weighted harmonic mean:

$$H_n = \frac{\sum_{i=1}^{n} n_i}{\sum_{i=1}^{n} n_i \frac{1}{m_i}} = \frac{N}{\sum_{i=1}^{n} n_i \frac{1}{m_i}} = \frac{1}{\sum_{i=1}^{n} x_i \frac{1}{m_i}}$$

Note that the use of a weighted average does not change the QM ≥ AM ≥ GM ≥ HM relationship, as the weighting factor can be taken out and replaced with an individual listing of the combined data points to give the simpler form of the average. An example for illustration:

$$M_n = \frac{\sum_{i=1}^{n} n_i m_i}{N} = \frac{3(5) + 4(6)}{7} = \frac{5+5+5+6+6+6+6}{7}$$

$$Q_n = \sqrt{\frac{\sum_{i=1}^{n} n_i m_i^2}{\sum_{i=1}^{n} n_i}} = \sqrt{\frac{2(3)^2 + 3(4)^2}{5}} = \sqrt{\frac{3^2 + 3^2 + 4^2 + 4^2 + 4^2}{5}}$$

Treating Data Points as Ratio Quantities:

One can keep track of units in all the different types of averages. Rather than averaging test scores as numbers, treat them as a ratio, *i.e.*, grade per person. If a student gets a grade of 80, then this is thought of as 80/person.

$$M_n = \frac{\sum_{i=1}^{n} n_i m_i}{\sum_{i=1}^{n} n_i} = \frac{\sum_{i=1}^{n} (\cancel{\#\ people}) \frac{grade}{person}}{\cancel{total\ \#\ of\ people}} = units\ of\ \frac{grade}{person}$$

Some averaged quantities are normally thought of as ratios; such as a speed in miles/hour or price in dollars/pound.

$$M_n = \frac{\sum_{i=1}^{n} n_i m_i}{\sum_{i=1}^{n} n_i} = \frac{\sum_{i=1}^{n}(\cancel{\#\ pounds})\dfrac{dollars}{pound}}{\cancel{total\ \#\ of\ pounds}} = units\ of\ \frac{dollars}{pound}$$

Here is the unit analysis for the harmonic mean of test grades thought of as grade/person.

$$\frac{N}{\sum_{i=1}^{n} n_i \frac{1}{m_i}} = \frac{total\ \#\ of\ people}{(\#people)\dfrac{1}{\dfrac{grade}{person}}} = \frac{total\ \#\ of\ people}{(\#people)\dfrac{person}{grade}}$$

$$\frac{N}{\sum_{i=1}^{n} n_i \frac{1}{m_i}} = \frac{people}{\dfrac{(people)^2}{grade}} = \frac{grade}{person}$$

Note that a weighted average can be based on differing amounts of either the numerator quantity or denominator quantity of the ratio being averaged. One can determine an average price for differing numbers of pounds at differing prices in dollars per pound. This is a weighted average based on differing amounts of the denominator quantity (*e.g.*, pounds in dollars/pound). Conversely, a weighted average price can be obtained for differing amounts of dollars spent at differing prices; *i.e.*, an average based on differing amounts of the price's numerator quantity (*e.g.*, dollars in dollars/pound). For a weighted average of differing values present in differing amounts of the denominator quantity, use the weighted arithmetic mean:

$$M_n = \frac{\sum_{i=1}^{n} n_i m_i}{\sum_{i=1}^{n} n_i} = \frac{(\#\ pounds)_i\left(\dfrac{dollars}{pound}\right)_i}{total\ \#\ of\ pounds} = \left(\frac{dollars}{pound}\right)_{average}$$

For a weighted average based on different amounts of the numerator quantity, use the weighted harmonic mean:

$$H_n = \frac{\sum_{i=1}^{n} n_i}{\sum_{i=1}^{n} n_i \frac{1}{m_i}} = \frac{total\ \$\ spent}{(\$\ at\ price)_i\ \dfrac{1}{\left(\dfrac{\$\ price}{pound}\right)_i}}$$

$$H_n = \frac{total\ \$}{(\$)_i\left(\dfrac{pounds}{\$}\right)_i} = \left(\frac{\$}{pound}\right)_{average}$$

When to use the different types of averages:

The quadratic mean is used when both positive and negative numbers must be considered based on their magnitude. The standard deviation for a population is a quadratic mean of deviations from the arithmetic mean. The quadratic mean is an important concept when considering alternating current, though it usually applied continuously with calculus.

The arithmetic mean yields the central value of a finite arithmetic series (*i.e.*, mean = median). We will designate the arithmetic mean (AM) as M_n for the remainder of this paper. The arithmetic mean of the finite sequence 3, 6, 9, 12, 15 is equal to 9 ($M_n = 9$). If there are an even number of terms in a finite arithmetic sequence, then the arithmetic mean is equal to the arithmetic mean of the central two terms. For instance, the arithmetic mean of the finite sequence 3, 6, 9, 12 is equal to the arithmetic mean of 6 & 9 (*i.e.*, 7.5). When the arithmetic mean (M_n) is multiplied by the total number of data points (N), one gets the total sum of all the data points.

$$M_n = \frac{\sum_{i=1}^{n} n_i m_i}{\sum_{i=1}^{n} n_i} = \frac{\sum_{i=1}^{n} n_i m_i}{N} \qquad NM_n = \sum_{i=1}^{n} n_i m_i$$

The geometric mean yields the central term of a finite geometric series (*i.e.*, mean = median). For instance, the geometric mean of the finite geometric sequence 3, 9, 27 81, 243 is equal to 27. If there are an even number of terms in a finite geometric series, then the geometric mean is equal to the geometric mean of the central two terms. For instance, the geometric mean of the finite sequence 3, 9, 27, 81 is equal to the geometric men of 9 & 27 (*i.e.*, $\sqrt{(9)(27)} = \sqrt{243} \approx 15.6$). The geometric mean should be used for data that accrues geometrically, such as the average annual change of an investment fund. Negative numbers can't be used in geometric means. The value of terms to be averaged geometrically should all be positive. One must convert a series containing negative terms to all positive values, possibly by converting the terms from values representing change to values representing remaining amount. For instance, an investment that gains 30% in year one, loses 20% in year two, and finally gains 15% in year three should be collated as 1.3, 0.80, 1.15 and not as 30, -20, 15. The geometric mean of 1.3, 0.80, 1.15 equals $\{(1.3)(0.80)(1.15)\}^{\frac{1}{3}} = (1.196)^{\frac{1}{3}} \approx 1.0615$ (average annual interest rate of approximately 6.15% yielding 19.6% gain after three years).

The harmonic mean yields the "full mediant" of a finite sequence of values wherein the numerators of all the values in the series are made identical. The full mediant of a finite sequence of terms is obtained by dividing the sum of the all the numerators by the sum of the all the denominators. Consider:

$$\frac{2}{3}, \frac{2}{5}, \frac{2}{9}, \frac{2}{10}, \frac{2}{11}, \frac{2}{11}$$

$$HM = \frac{6}{\frac{3}{2} + \frac{5}{2} + \frac{9}{2} + \frac{10}{2} + \frac{11}{2} + \frac{11}{2}} = \frac{6(2)}{3 + 5 + 9 + 10 + 11 + 11}$$

Here is another example:

$$3,5,7,\frac{2}{3},11=\frac{2}{\frac{2}{3}},\frac{2}{\frac{2}{5}},\frac{2}{\frac{2}{7}},\frac{2}{3},\frac{2}{\frac{2}{11}}$$

$$HM=\frac{5}{\frac{2/3}{2}+\frac{2/5}{2}+\frac{2/7}{2}+\frac{3}{2}+\frac{2/11}{2}}=\frac{5(2)}{\frac{2}{3}+\frac{2}{5}+\frac{2}{7}+3+\frac{2}{11}}$$

Note that when the denominators form an arithmetic series, assuming a finite sequence of values wherein all the numerators are equal, then the harmonic mean is equal to the central term in the series (*i.e.*, mean = median) or, in the case where there are an even number of terms, the mediant (equivalent to the harmonic mean) of the central two terms.

$$\frac{1}{3},\frac{1}{5},\frac{1}{7},\frac{1}{9},\frac{1}{11}$$

$$HM=\frac{5}{3+5+7+9+11}=\frac{1}{7}$$

$$\frac{3}{5},\frac{3}{8},\frac{3}{11},\frac{3}{14}$$

$$HM=\frac{4}{\frac{5}{3}+\frac{8}{3}+\frac{11}{3}+\frac{14}{3}}=\frac{4(3)}{5+8+11+14}=\frac{12}{38}=\frac{6}{19}=\frac{3+3}{8+11}$$

As previously shown, the weighted harmonic mean is used when one is calculating a weighted average with weighting by differing amounts of the numerator quantity of the ratio values being averaged. Note that using a weighted harmonic mean in this way provides a value which is identical to the arithmetic mean that would have been obtained had the weighting been done by equivalent amounts (*i.e.*, equivalent to the amount, not the same value) of the denominator quantity of the ratio value being averaged.

A speed in miles per hour is an example of a ratio value. If one wants a weighted average for different speeds (miles per hour) traveled for differing amounts of time (hours – the denominator of the ratio value being averaged), then one uses a weighted arithmetic mean.

$$t_i = time\ at\ speed\ i \quad s_i = speed\ i \quad T = total\ time = \sum_{i=1}^{n} t_i$$

$$M_n = \frac{\sum_{i=1}^{n} t_i s_i}{\sum_{i=1}^{n} t_i} = \sum_{i=1}^{n} \frac{t_i s_i}{T}$$

The harmonic mean is used when the ratio value is weighted by variable amounts of the ratio's numerator quantity (distance in miles for speed in miles/hour).

$$l_i = distance\ at\ speed\ i \quad s_i = speed\ i \quad L = total\ distance = \sum_{i=1}^{n} l_i$$

$$H_n = \frac{\sum_{i=1}^{n} l_i}{\sum_{i=1}^{n} l_i \frac{1}{s_i}} = \frac{L}{\sum_{i=1}^{n} l_i \frac{1}{s_i}}$$

Note that this harmonic mean is equal to the arithmetic mean that would be obtained if the weighting was in times (hours) corresponding to the distances (miles).

If the weightings based on a denominator quantity are all equal, then the weightings can be replaced as division by n (number of data points) yielding the simple arithmetic mean.

$$M_n = \frac{\sum_{i=1}^{n} t_i s_i}{\sum_{i=1}^{n} t_i} = \frac{t_i}{n t_i} \sum_{i=1}^{n} s_i = \frac{1}{n} \sum_{i=1}^{n} s_i$$

If the weightings based on a numerator quantity are all equal, then the weightings can be replaced by a numerator equal to "n" (number of data points) yielding the simple harmonic mean.

$$H_n = \frac{\sum_{i=1}^{n} l_i}{\sum_{i=1}^{n} l_i \frac{1}{s_i}} = \frac{n l_i}{l_i \sum_{i=1}^{n} \frac{1}{s_i}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{s_i}}$$

Measures of Dispersity:

The variance and the standard deviation are two common measures of the dispersity of a data set.

$$s = \sqrt{\frac{\sum_{i=1}^{n} n_i (M_n - m_i)^2}{\sum_{i=1}^{n} n_i}} = \sqrt{\frac{\sum_{i=1}^{n} n_i (M_n - m_i)^2}{N}} = stand.dev.$$

$$s^2 = \frac{\sum_{i=1}^{n} n_i (M_n - m_i)^2}{\sum_{i=1}^{n} n_i} = \frac{\sum_{i=1}^{n} n_i (M_n - m_i)^2}{N} = variance$$

Another measure of dispersity known as the "Polydispersity Index" is used in polymer science. When a polymer is produced, there will be a range of molecular weights (MW's) produced. The number average molecular weight of the polymer can be calculated as:

$$n_i = number\ of\ molecules\ (moles) with\ GMW\ of\ m_i$$

$$N = total\ number\ of\ molecules\ (moles)$$

$$x_{i,n} = number\ (mole)\ fraction\ of\ fraction = \frac{n_i}{N}$$

$$M_n = \frac{\sum_{i=1}^{n} n_i m_i}{\sum_{i=1}^{n} n_i} = \frac{\sum_{i=1}^{n} n_i m_i}{N} = \sum_{i=1}^{n} x_{i,n} m_i$$

The number average molecular weight is a traditional weighted arithmetic mean. One can also calculate a weight average molecular weight as M_w:

$$NM_n = \sum_{i=1}^{n} n_i m_i = total\ weight\ of\ polymer$$

$$M_w = \frac{\frac{\sum_{i=1}^{n} n_i m_i^2}{N}}{M_n} = \frac{\sum_{i=1}^{n} n_i m_i^2}{NM_n} = \frac{\frac{\sum_{i=1}^{n} n_i m_i^2}{N}}{\frac{\sum_{i=1}^{n} n_i m_i}{N}} = \frac{\sum_{i=1}^{n} n_i m_i^2}{\sum_{i=1}^{n} n_i m_i} = \frac{\sum_{i=1}^{n} x_i m_i^2}{\sum_{i=1}^{n} x_i m_i}$$

$$M_w = \frac{\sum_{i=1}^{n} n_i m_i m_i}{NM_n} = \frac{\sum_{i=1}^{n} (weight\ of\ fraction) m_i}{(total\ weight\ of\ polymer)}$$

$$M_w = \sum_{i=1}^{n} x_{i,w} m_i \qquad x_{i,w} = weight\ fraction$$

The weight average molecular weight is an average of the molecular weights weighted by their weight fraction rather than the number fraction. Note that this type of "extended" average can be calculated for any ratio quantity. Consider a similar set of weighted averages for a set of apple prices in dollars per pound for differing weights of apples.

$$p_i = price\ of\ apples\ for\ fraction\ "i"\ in\ dollars/pound$$

$$w_i = weight\ of\ apples\ for\ fraction\ i\ in\ pounds$$

$$w_i p_i = dollars\ spent\ to\ buy\ w_i\ pounds\ at\ price\ p_i$$

$$\sum_{i=1}^{n} w_i p_i = total\ dollars\ spent = WM_n$$

$$\sum_{i=1}^{n} w_i = total\ weight\ of\ apples = W$$

$$M_n = \frac{\sum_{i=1}^{n} w_i p_i}{\sum_{i=1}^{n} w_i} = \frac{\sum_{i=1}^{n} w_i p_i}{W} = \sum_{i=1}^{n} x_{i,n} p_i = \frac{total\ dollars\ spent}{total\ weight}$$

$$x_{i,n} = weight\ fraction\ of\ fraction$$

The weight average for this type of data set would be:

$$M_w = \frac{\sum_{i=1}^{n} w_i p_i p_i}{WM_n} = \frac{\sum_{i=1}^{n}(price\ of\ fraction) p_i}{(total\ price)} = \sum_{i=1}^{n} x_{i,w} p_i$$

$$x_{i,w} = price\ fraction\ of\ fraction$$

The weight average apple price is an average of the apple prices weighted by their price fraction rather than their weight fraction.

<u>Relationship of Mn & Mw to the variance:</u>

$$s^2 = \frac{\sum_{i=1}^{n} n_i (M_n - m_i)^2}{\sum_{i=1}^{n} n_i} = \frac{\sum_{i=1}^{n} n_i (M_n - m_i)^2}{N} = variance$$

$$\frac{\sum_{i=1}^{n} n_i (M_n - m_i)^2}{\sum_{i=1}^{n} n_i} = \frac{\sum_{i=1}^{n} n_i (M_n^2 - 2M_n m_i + m_i^2)}{N}$$

$$s^2 = \frac{\sum_{i=1}^{n} n_i M_n^2}{N} - 2\frac{\sum_{i=1}^{n} n_i m_i M_n}{N} + \frac{\sum_{i=1}^{n} n_i m_i^2}{N}$$

$$\frac{s^2}{M_n^2} = \frac{\sum_{i=1}^{n} n_i M_n^2}{N M_n^2} - 2\frac{\sum_{i=1}^{n} n_i m_i M_n}{N M_n^2} + \frac{\sum_{i=1}^{n} n_i m_i^2}{N M_n^2}$$

$$\left(\frac{s}{M_n}\right)^2 = \frac{\sum_{i=1}^{n} n_i}{N} - 2\frac{\sum_{i=1}^{n} n_i m_i}{N M_n} + \frac{\sum_{i=1}^{n} n_i m_i^2}{N M_n M_n}$$

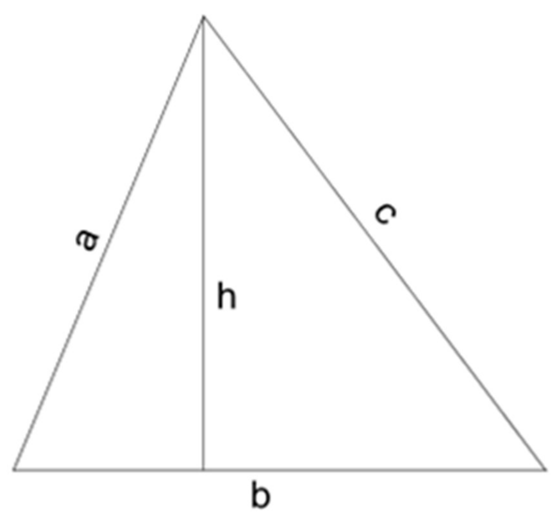
$$\left(\frac{s}{M_n}\right)^2 = 1 - 2 + \frac{M_w}{M_n} = \frac{M_w}{M_n} - 1$$

$$\left(\frac{s}{M_n}\right)^2 = \frac{M_w}{M_n} - 1$$

$$\frac{M_w}{M_n} = polydispersity\ index$$

Appendix VI: The Analytic Triangle:

See the diagram below.



$$h = \sqrt{a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2} = \sqrt{c^2 - \left(\frac{c^2 + b^2 - a^2}{2b}\right)^2}$$

$$\sqrt{a^2 - h^2} + \sqrt{c^2 - h^2} = b$$

The calculation of h from this equation yields, of course, the same result as calculation directly from the triangle.

$$\left\{\sqrt{a^2 - h^2}\right\}^2 = \left\{b - \sqrt{c^2 - h^2}\right\}^2$$

$$a^2 - h^2 = b^2 - 2b\sqrt{c^2 - h^2} + c^2 - h^2$$

$$\left(\frac{a^2 - c^2 - b^2}{-2b}\right)^2 = c^2 - h^2$$

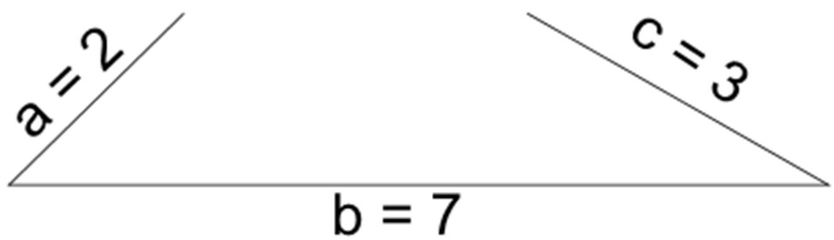
$$c^2 - \left(\frac{c^2 + b^2 - a^2}{2b}\right)^2 = h^2$$

$$h = \sqrt{c^2 - \left(\frac{c^2 + b^2 - a^2}{2b}\right)^2}$$

A generalized triangle with real or complex leg lengths can be associated with this equation.

$$\sqrt{a^2 - h^2} + \sqrt{c^2 - h^2} = b \qquad a, b, c \text{ are constants} \qquad h \text{ is the variable}$$

Triangles with “impossible” leg lengths are useful for providing a geometric visualization.

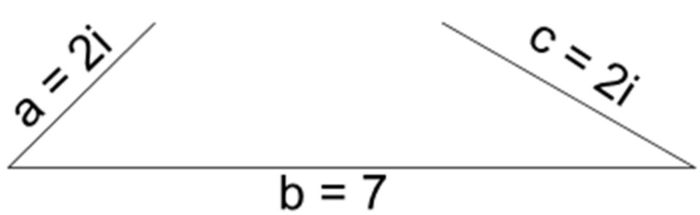


$$h = \sqrt{9 - \left(\frac{9 + 49 - 4}{2(7)}\right)^2} = \sqrt{9 - \left(\frac{54}{14}\right)^2} = \sqrt{\frac{9(196)}{196} - \frac{2916}{196}} = \sqrt{\frac{1152}{196}}\,i$$

$$\sqrt{2^2 - \left(\sqrt{\frac{1152}{196}}\,i\right)^2} + \sqrt{3^2 - \left(\sqrt{\frac{1152}{196}}\,i\right)^2} = 7$$

$$\sqrt{4 + \frac{1152}{196}} + \sqrt{9 + \frac{1152}{196}} = 7$$

Triangles with imaginary leg lengths are also fine.

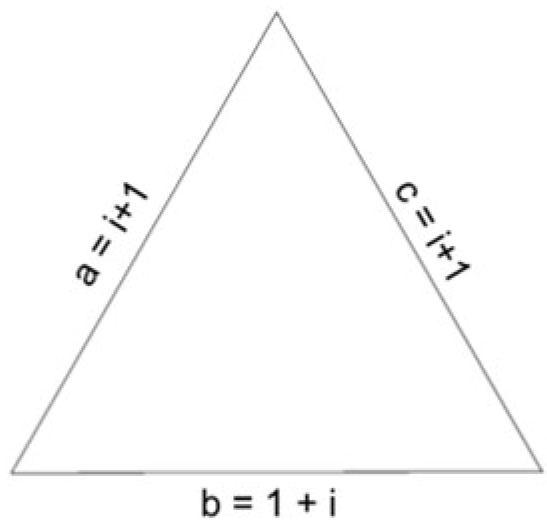


$$h = \sqrt{(2i)^2 - \left(\frac{(2i)^2 + 49 - (2i)^2}{2(7)}\right)^2} = \sqrt{-4 - \left(\frac{-4 + 49 + 4}{14}\right)^2} = \sqrt{\frac{3185}{196}}i$$

$$\sqrt{(2i)^2 - \left(\sqrt{\frac{3185}{196}}i\right)^2} + \sqrt{(2i)^2 - \left(\sqrt{\frac{3185}{196}}i\right)^2} = 7$$

$$\sqrt{-4 + \frac{3185}{196}} + \sqrt{-4 + \frac{3185}{196}} = 7$$

Likewise, legs with complex leg lengths are also fine.



$$h = \sqrt{(1+i)^2 - \left(\frac{(1+i)^2 + (1+i)^2 - (1+i)^2}{2(1+i)}\right)^2} = \sqrt{2i - \left(\frac{2i}{2+2i}\right)^2}$$

$$h = \sqrt{2i - \frac{-4}{8i}} = \sqrt{2i + \frac{1}{2i}} = \sqrt{2i - \frac{i}{2}} = \sqrt{\frac{3}{2}i} \qquad \sqrt{i} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$h = \sqrt{\frac{3}{2}i} = \sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}i$$

$$\sqrt{(1+i)^2 - \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}i\right)^2} + \sqrt{(1+i)^2 - \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}i\right)^2} = 1 + i$$

$$\sqrt{(1+i)^2-\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}i\right)^2}=\sqrt{2i-\frac{3}{2}i}=\sqrt{\frac{1}{2}i}=\sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=\frac{1}{2}+\frac{1}{2}i$$

$$\sqrt{(1+i)^2-\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}i\right)^2}+\sqrt{(1+i)^2-\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}i\right)^2}=1+i$$

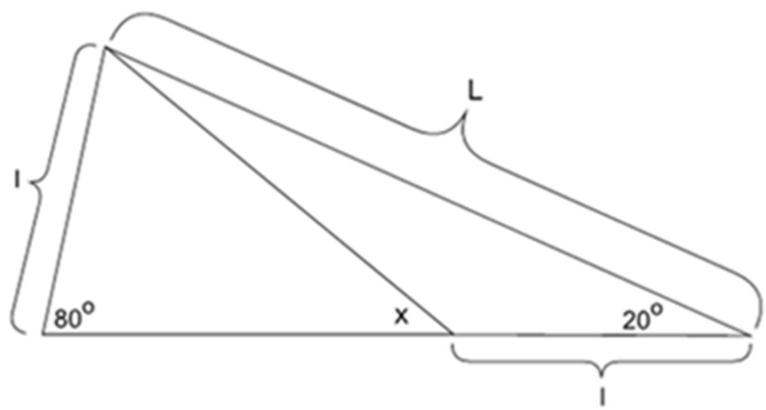
Triangles with complex leg lengths have a complex area. The area of the (1 + i), (1 + i), (1 + i) triangle is:

$$Area = A = \frac{hb}{2} = \frac{\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}i\right)(1+i)}{2} = \frac{\sqrt{3}}{2}i$$

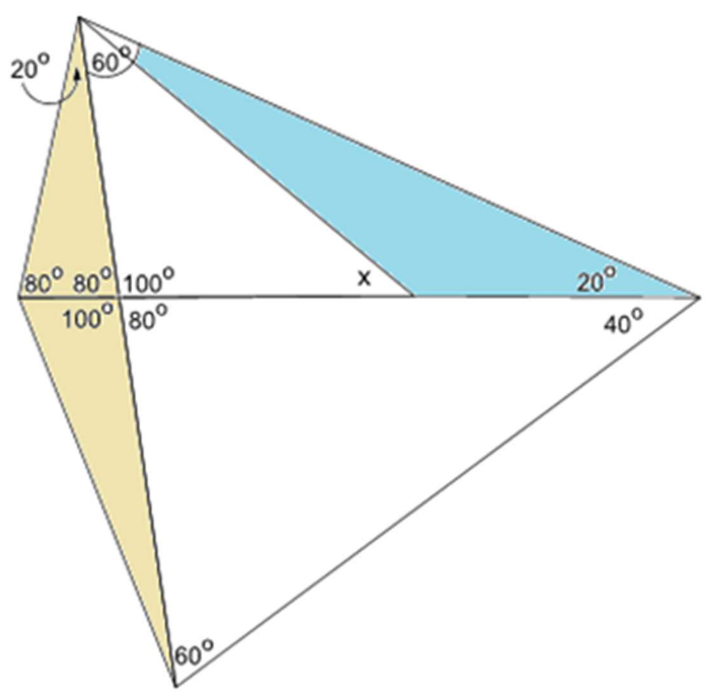
From an analytic point of view, there is no problem using triangles with complex leg lengths.

Appendix VII: Triangle Puzzles:

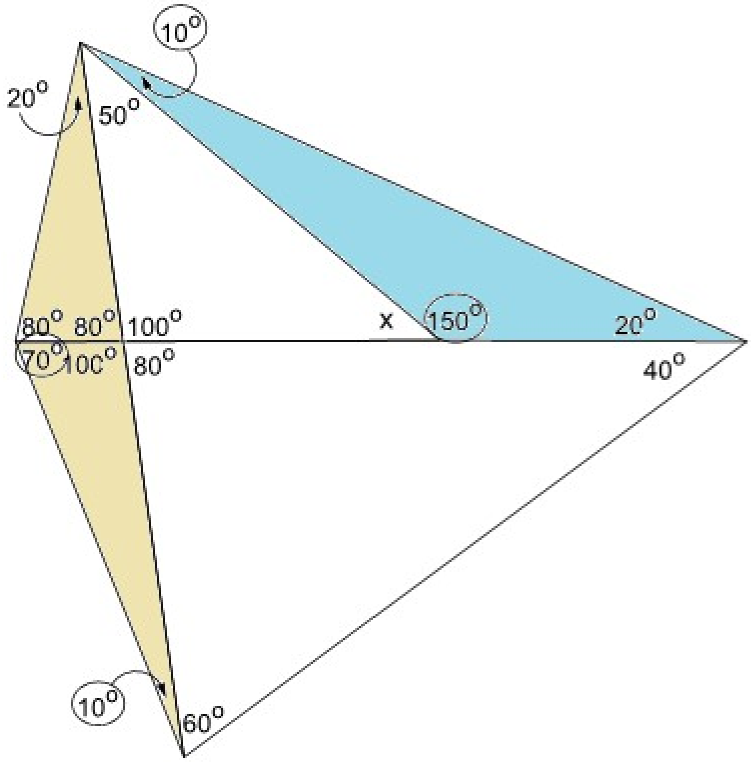
Puzzle 1: Find the angle x. This is an example of a constructive solution to a puzzle.



Create an equilateral triangle with sides of length "L". Identify and mark all the known angles, and construct a new triangle identical to the one filled with light blue (identical by side-angle-side).

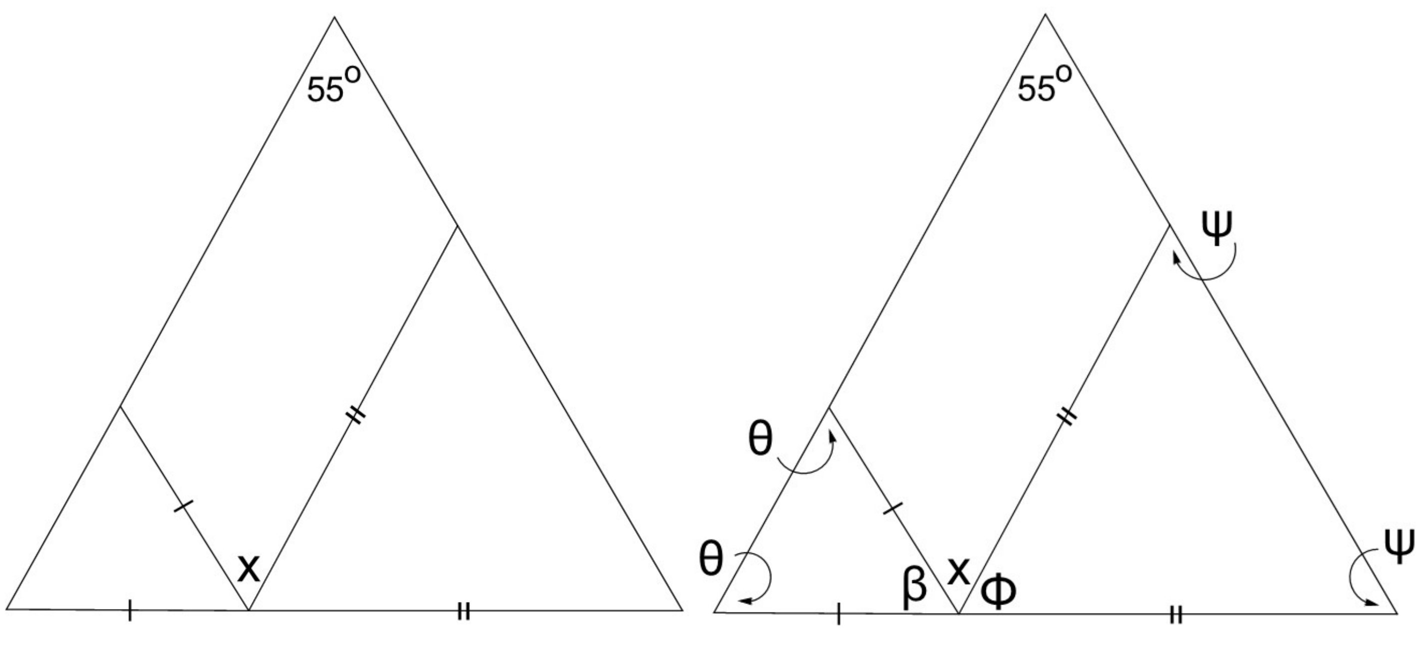


Note that we can now identify two legs of the original triangle as being of identical length "L" based on fact that two of the vertices have identical angles of 80°. This allows the following new angle identifications (circled angles).



Now the angle "x" can be identified as 30°.

Puzzle 2: Find the angle x. This is an example of a simple combined geometry and algebra problem (problem as typically presented is on the left; angles identified on right). The only "trick" is to recognize the values of the angles θ, ψ, β, ϕ.



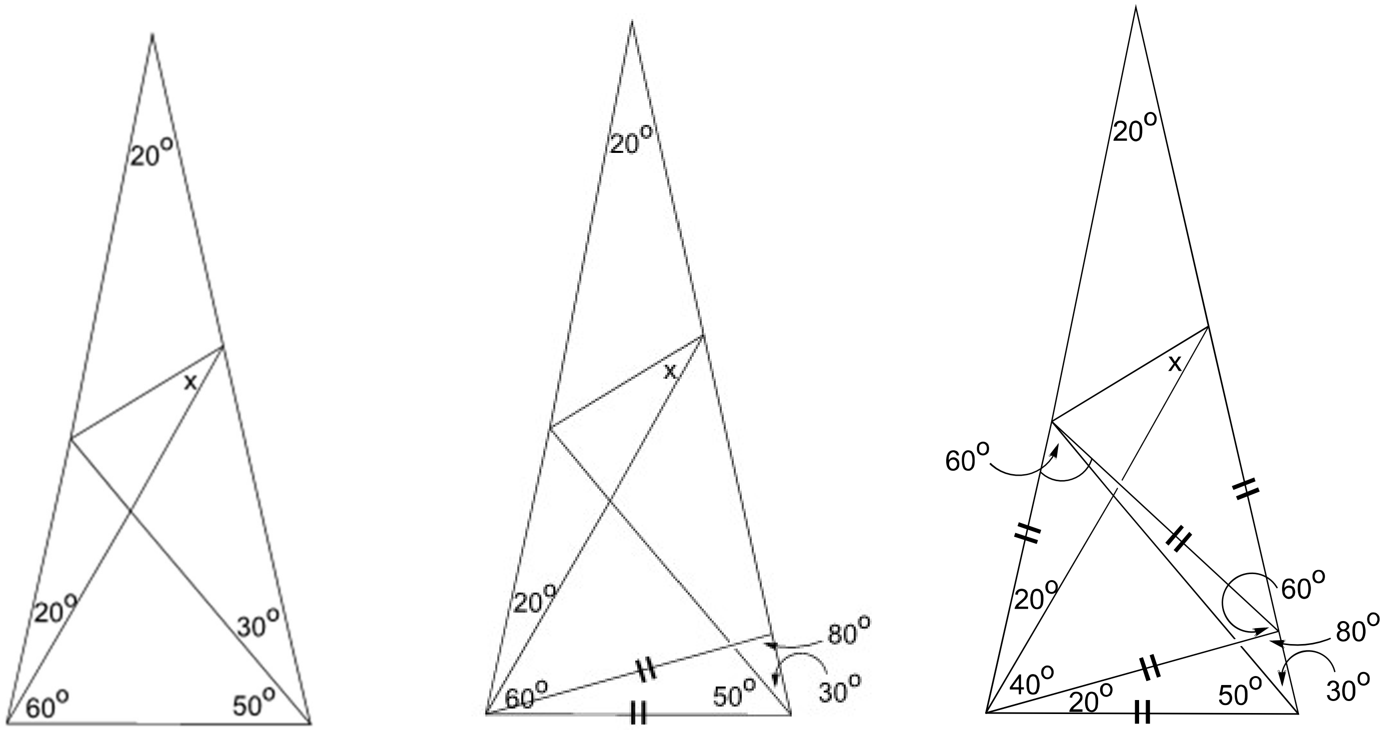
$$\theta + \psi + 55° = 180° \qquad \theta + \psi = 125°$$

$$2\theta + \beta = 180° \qquad 2\psi + \phi = 180° \qquad x + \beta + \phi = 180°$$

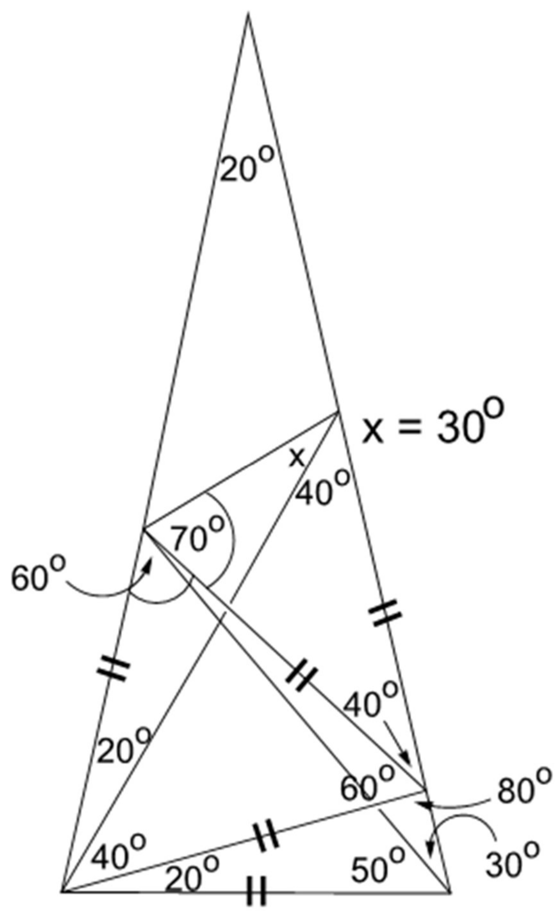
$$x + 180° - 2\theta + 180° - 2\psi + 180° = 180°$$

$$x = 2(\theta + \psi) - 180 = 2(125°) - 180° = 70°$$

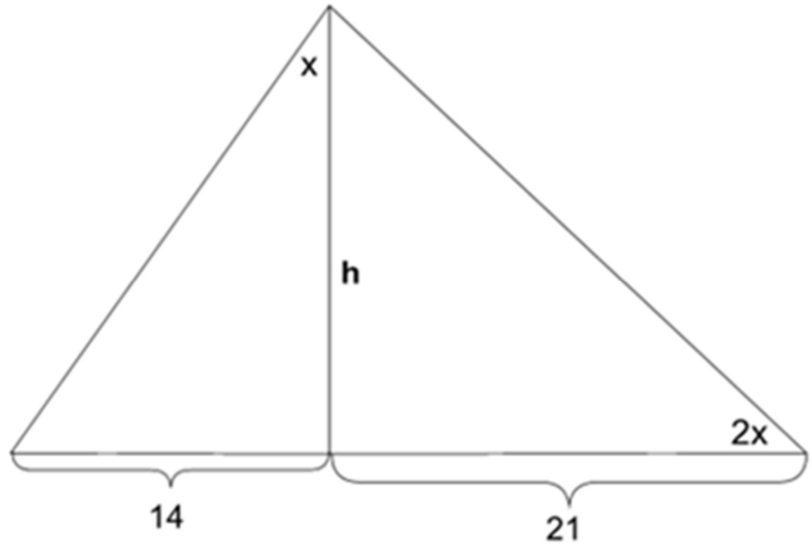
Puzzle 3: What is the unknown angle? This is a problem known as Langley's Adventitious Angle. The leftmost triangle is how the problem is typically presented. The problem is adventitious in that it is not solvable unless the triangle's angles and lengths are properly set. The central and rightmost triangles show how some additional angles can be identified via construction of new angles by inserting line segments of length equal to the length of the triangle's base.



The final solution is given below. Note that the term adventitious is being used ironically in that the angles, lengths, etc. are not accidental, but rather they must be set exactly to certain whole numbers and equivalent length sections in order for the problem to be solvable.



Puzzle 4: What is the area of the triangle? This is a problem that uses simple trigonometry.

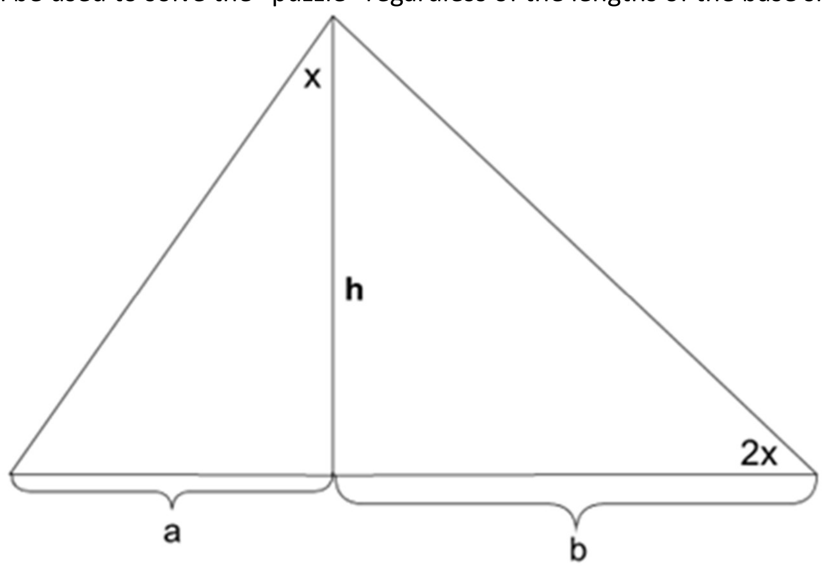


$$Tan(x) = \frac{14}{h} \qquad Tan(2x) = \frac{2Tan(x)}{1 - Tan^2(x)} = \frac{\frac{28}{h}}{1 - \frac{196}{h^2}}$$

$$Tan(2x) = \frac{28h}{h^2 - 196} = \frac{h}{21} \qquad 588h = h^3 - 196h \qquad h^3 = 784h \qquad h^2 = 784$$

$$h = 28 \qquad A = \frac{(28)(35)}{2} = 490$$

Note that this method can be used to solve the "puzzle" regardless of the lengths of the base side segments.

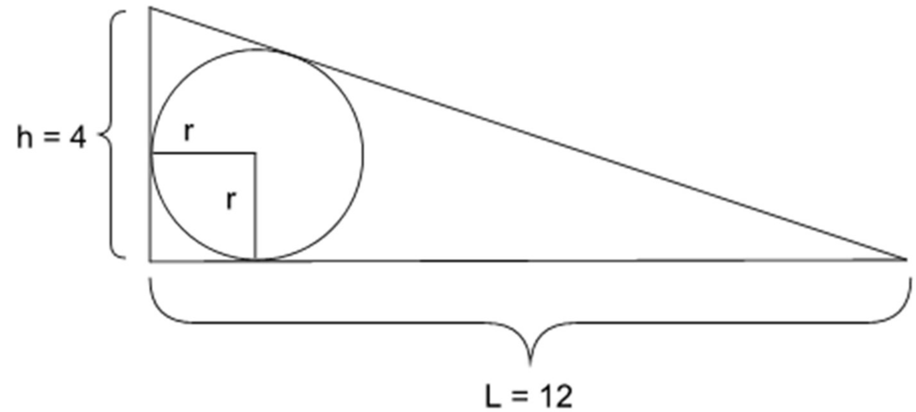


$$Tan(x) = \frac{a}{h} \qquad Tan(2x) = \frac{2Tan(x)}{1 - Tan^2(x)} = \frac{\frac{2a}{h}}{1 - \frac{a^2}{h^2}}$$

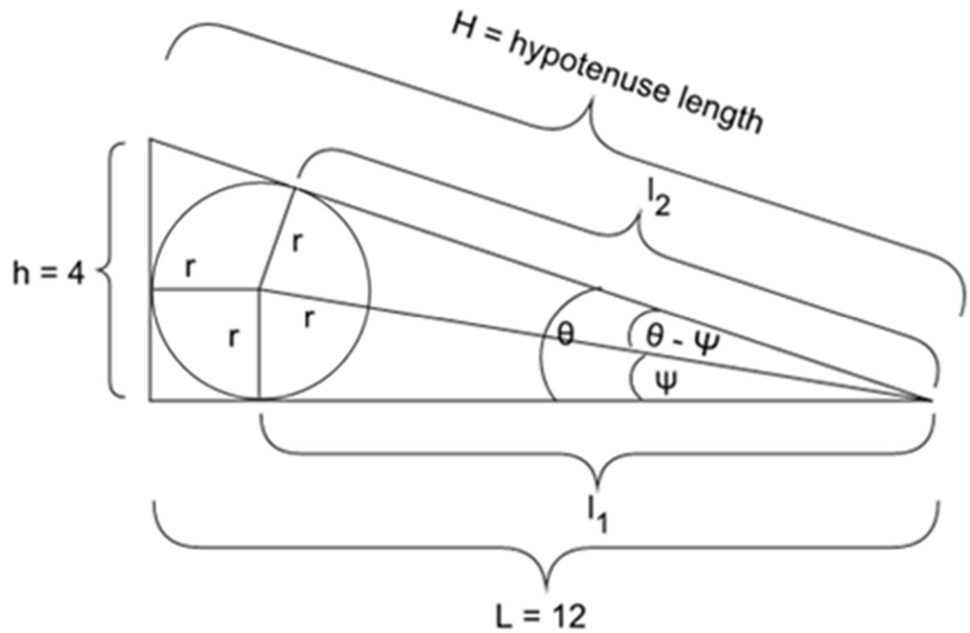
$$Tan(2x) = \frac{2ah}{h^2 - a^2} = \frac{h}{b} \qquad 2abh = h^3 - a^2h \qquad h^3 = h(2ab + a^2) \qquad h^2 = 2ab + a^2$$

$$h = \sqrt{2ab + a^2} \qquad A = \frac{(a + b)\sqrt{2ab + a^2}}{2}$$

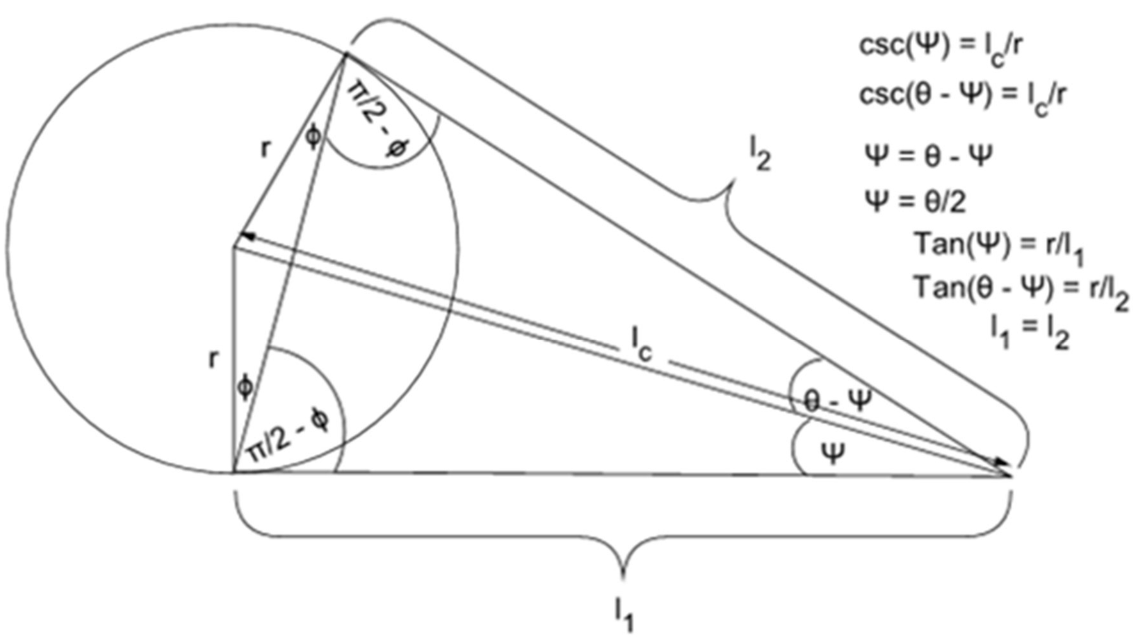
Puzzle 5: What is the radius of the circle? What percentage of the area of the triangle is taken up by the circle?



One can diagram the problem as follows.



Note that l_1 and l_2 are equal.



Solve the problem in general first.

$$Tan(\psi) = \frac{r}{l} = Tan\left(\frac{\theta}{2}\right) = \frac{Sin(\theta)}{1+\cos(\theta)} = \frac{\frac{h}{H}}{1+\frac{L}{H}} = \frac{h}{H+L} = \frac{h}{\sqrt{h^2+L^2}+L}$$

$$r = \frac{hl}{H+L} \qquad L = l + r \qquad l = L - r \qquad l_1 = l_2 = l$$

$$r = \frac{h(L-r)}{H+L} = \frac{hL}{H+L} - \frac{hr}{H+L}$$

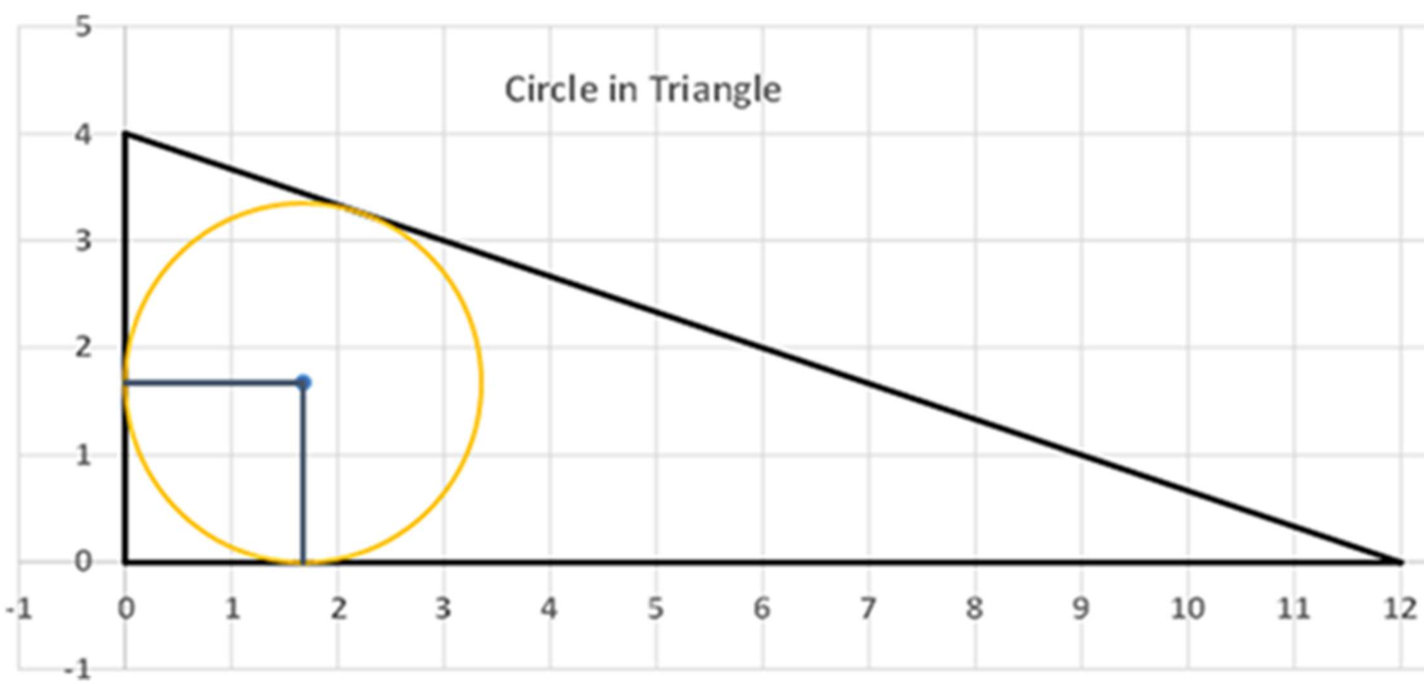
$$r + \frac{hr}{H+L} = r\left(1+\frac{h}{H+L}\right) = \frac{hL}{H+L}$$

$$r = \frac{\frac{hL}{H+L}}{1+\frac{h}{H+L}} = \frac{hL}{H+L+h} = \frac{(4)(12)}{\sqrt{160}+12+4} = 1.67544468$$

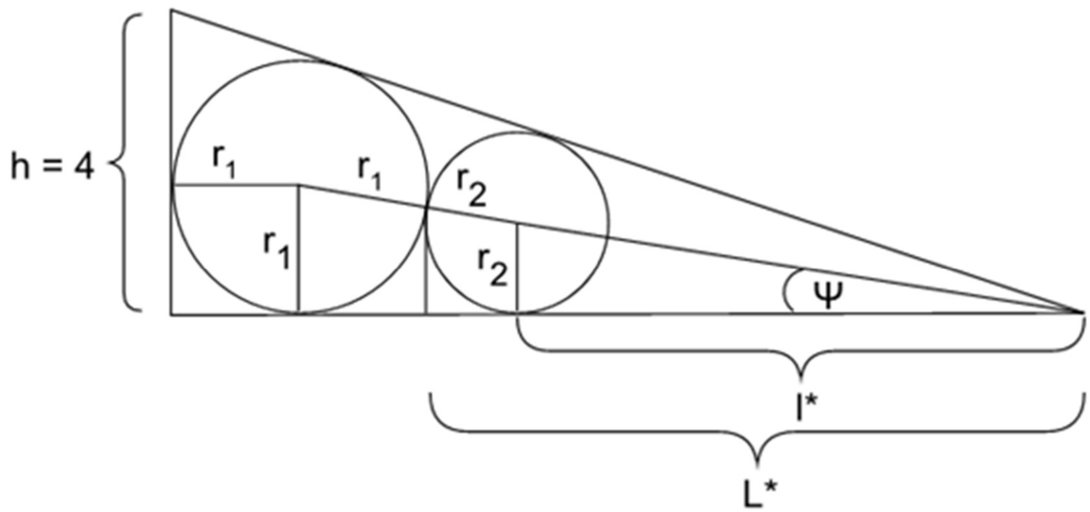
The area of the triangle is 24, so the % area consumed by the circle is:

$$\%\ Area\ Consumed = \frac{\pi r^2}{24}(100) = \frac{\pi(1.67544468)^2}{24}(100) = 36.745\%$$

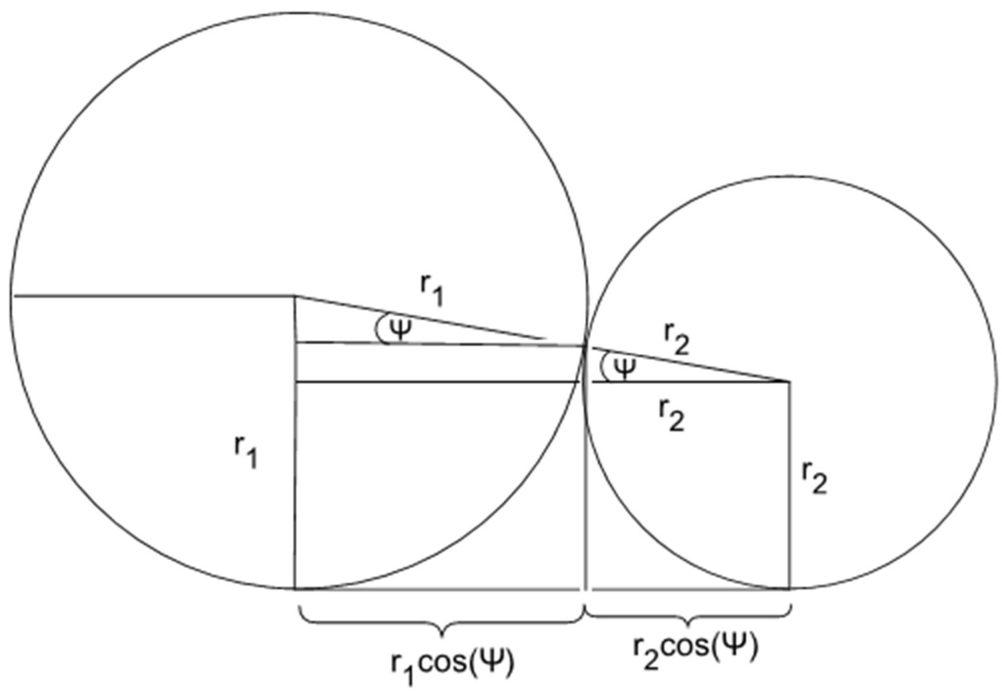
If the vertex of the right triangle's legs is set at (0, 0), then the center of the circle is at (r, r). See a graphic showing this circle in triangle below.



Next, we can extend the construction to another circle.



The calculation of r_2 is similar to r_1 except for the calculation of the new value for L*.



The new calculation is:

$$Tan(\psi) = \frac{r_2}{l^*} = Tan\left(\frac{\theta}{2}\right) = \frac{Sin(\theta)}{1+\cos(\theta)} = \frac{\frac{h}{H}}{1+\frac{L}{H}} = \frac{h}{H+L}$$

$$r_2 = \frac{hl^*}{H+L} \qquad L^* = L - r_1 - r_1\cos(\psi) \qquad l^* = L^* - r_2\cos(\psi)$$

$$r_2 = \frac{h\left\{L^* - r_2\cos\left(\frac{\theta}{2}\right)\right\}}{H+L} = \frac{h\{L - r_1 - r_1\cos(\psi) - r_2\cos(\psi)\}}{H+L}$$

$$r_2 + \frac{hr_2\cos\left(\frac{\theta}{2}\right)}{H+L} = r_2\left\{1 + \frac{h\cos\left(\frac{\theta}{2}\right)}{H+L}\right\} = \frac{h\left\{L - r_1 - r_1\cos\left(\frac{\theta}{2}\right)\right\}}{H+L}$$

$$r_2 = \frac{\dfrac{h\left\{L - r_1 - r_1\cos\left(\frac{\theta}{2}\right)\right\}}{H+L}}{1 + \dfrac{h\cos\left(\frac{\theta}{2}\right)}{H+L}} = \frac{h\left\{L - r_1 - r_1\cos\left(\frac{\theta}{2}\right)\right\}}{H + L + h\cos\left(\frac{\theta}{2}\right)}$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos(\theta)}{2}} = \sqrt{\frac{1 + \frac{12}{\sqrt{160}}}{2}} = 0.987087457$$

$$r_2 = \frac{h\left\{L - r_1 - r_1\cos\left(\frac{\theta}{2}\right)\right\}}{H + L + h\cos\left(\frac{\theta}{2}\right)} = \frac{4\{12 - 1.67544468 - (1.67544468)(0.987087457)\}}{\sqrt{160} + 12 + 4(0.987087457)}$$

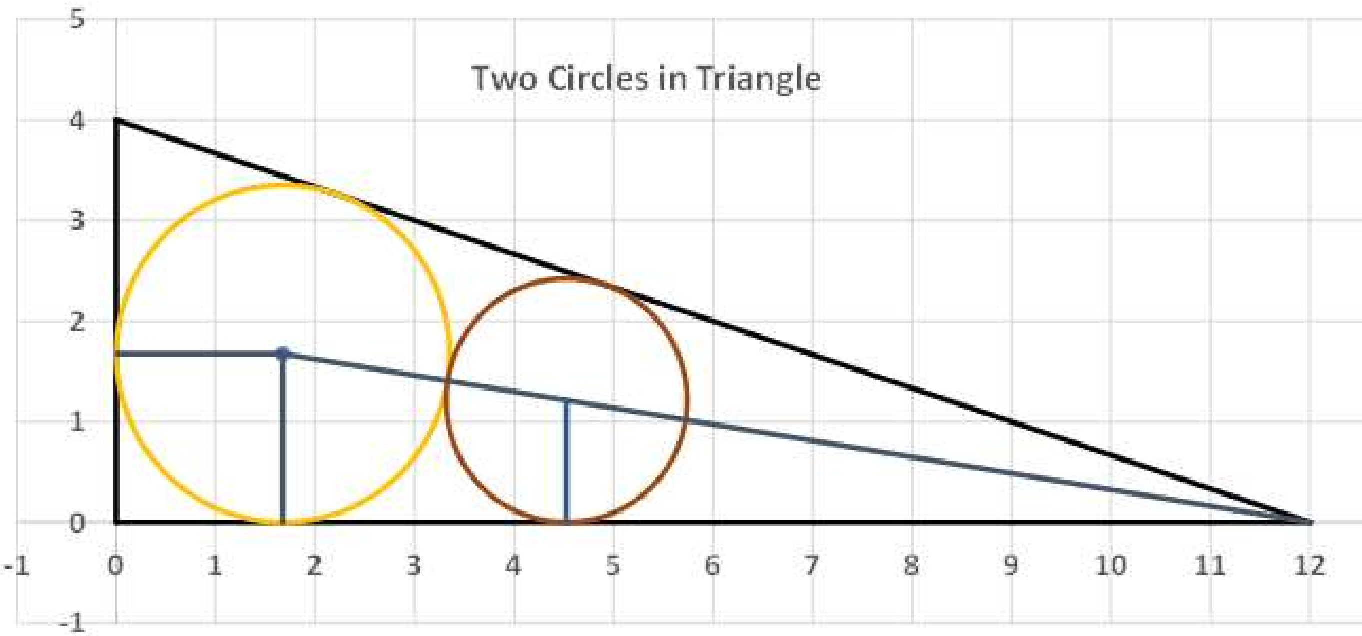
$$r_2 = 1.212799283$$

The center of the new circle (assuming the vertex of the legs of the right triangle is at 0, 0) has an x-value of:

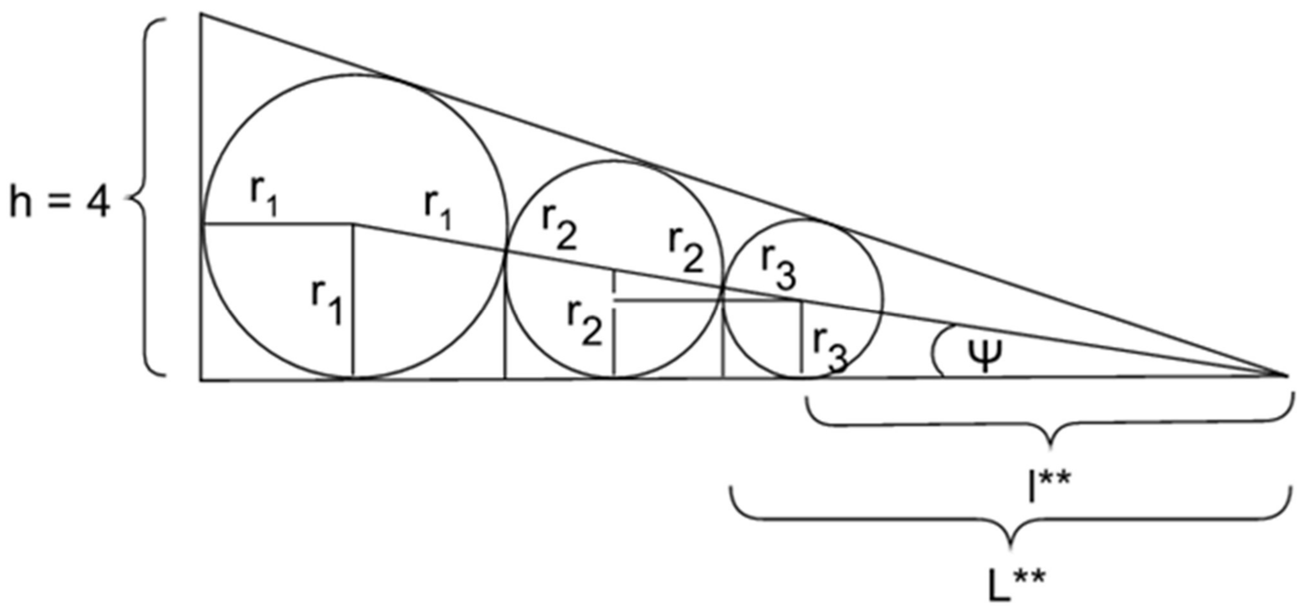
$$x_2 = r_1 + r_1\cos\left(\frac{\theta}{2}\right) + r_2\cos\left(\frac{\theta}{2}\right) = 4.526394069$$

$$y_2 = r_2 = 1.212799283$$

The graphic below illustrates the two “kissing” circles inside a right triangle.



This can be extended to a third circle.



$$Tan(\psi) = \frac{r_3}{l^{**}} = Tan\left(\frac{\theta}{2}\right) = \frac{Sin(\theta)}{1+\cos(\theta)} = \frac{\frac{h}{H}}{1+\frac{L}{H}} = \frac{h}{H+L}$$

$$r_3 = \frac{hl^{**}}{H+L} \qquad L^{**} = L - r_1 - r_1\cos(\psi) - 2r_2\cos(\psi) \qquad l^{**} = L^{**} - r_3\cos(\psi)$$

$$r_3 = \frac{h\{L - r_1 - r_1\cos(\psi) - 2r_2\cos(\psi) - r_3\cos(\psi)\}}{H+L}$$

$$r_3\left(1 + \frac{h\cos(\psi)}{H+L}\right) = \frac{h\{L - r_1 - r_1\cos(\psi) - 2r_2\cos(\psi)\}}{H+L}$$

$$r_3 = \frac{\frac{h\{L - r_1 - r_1\cos(\psi) - 2r_2\cos(\psi)\}}{H+L}}{1+\frac{\cos(\psi)}{H+L}} = \frac{h\left\{L - r_1 - r_1\cos\left(\frac{\theta}{2}\right) - 2r_2\cos\left(\frac{\theta}{2}\right)\right\}}{H+L+h\cos\left(\frac{\theta}{2}\right)}$$

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos(\theta)}{2}} = \sqrt{\frac{1+\frac{12}{\sqrt{160}}}{2}} = 0.987087457$$

$$r_3 = \frac{4(12 - 1.67544468 - (1.67544468)(0.987087457) - 2(1.212799283)(0.987087457)}{\sqrt{160} + 12 + (4)(0.987087457)}$$

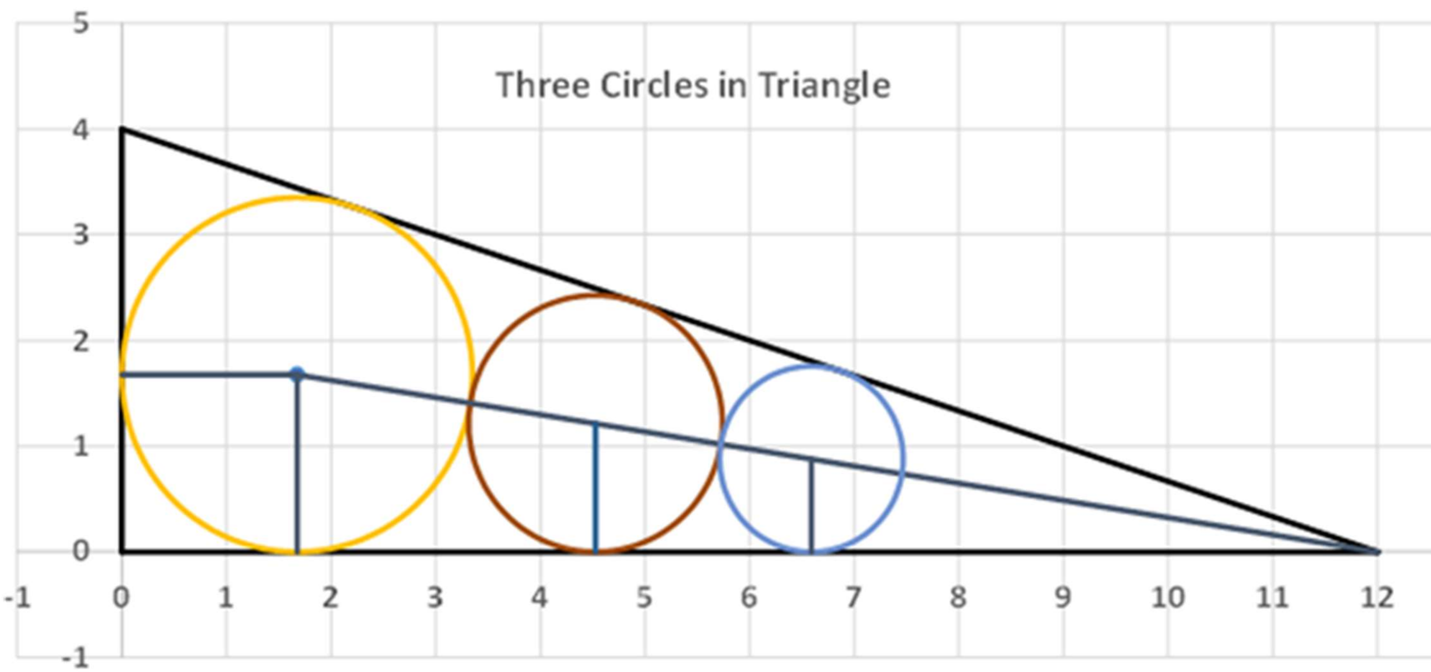
$$r_3 = \frac{25.10585951}{28.59746047} = 0.877905211$$

The coordinates of the center of the third circle are:

$$x_3 = r_1 + r_1\cos\left(\frac{\theta}{2}\right) + 2r_2\cos\left(\frac{\theta}{2}\right) + r_3\cos\left(\frac{\theta}{2}\right) = 6.590102251$$

$$y_3 = r_3 = 0.877905211$$

The graphic below illustrates the three “kissing” circles inside a right triangle.



The process can be extended systematically:

$$r_4 = \frac{hl^{***}}{H+L} \qquad l^{***} = L - r_1 - r_1\cos(\psi) - 2r_2\cos(\psi) - 2r_3\cos(\psi) - r_4\cos(\psi)$$

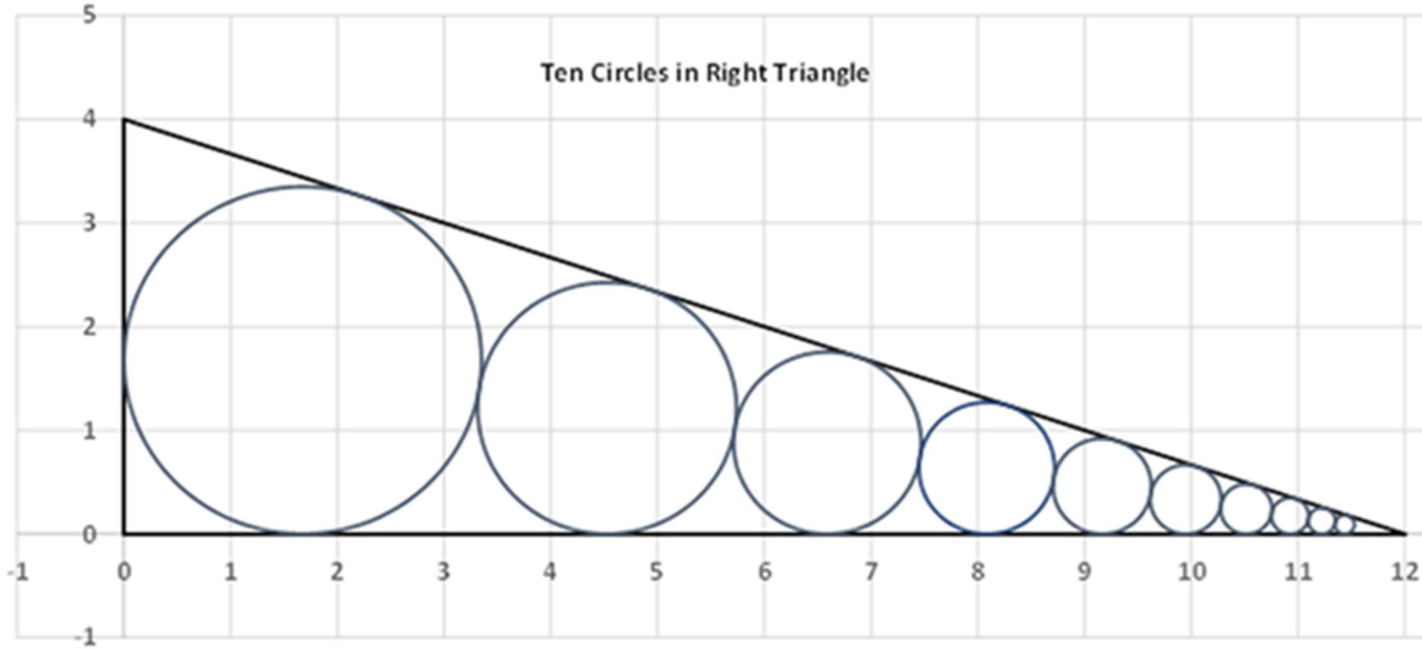
$$r_4 = \frac{h\left\{L - r_1 - r_1\cos\left(\frac{\theta}{2}\right) - 2r_2\cos\left(\frac{\theta}{2}\right) - 2r_3\cos\left(\frac{\theta}{2}\right)\right\}}{H + L + h\cos\left(\frac{\theta}{2}\right)}$$

$$x_4 = r_1 + r_1\cos\left(\frac{\theta}{2}\right) + 2r_2\cos\left(\frac{\theta}{2}\right) + 2r_3\cos\left(\frac{\theta}{2}\right) + r_4\cos\left(\frac{\theta}{2}\right)$$

$$r_n = \frac{h\left\{L - r_1 - r_1\cos\left(\frac{\theta}{2}\right) - 2r_2\cos\left(\frac{\theta}{2}\right) - 2r_3\cos\left(\frac{\theta}{2}\right) - \cdots - 2r_n\cos\left(\frac{\theta}{2}\right)\right\}}{H + L + h\cos\left(\frac{\theta}{2}\right)}$$

$$x_n = r_1 + r_1 \cos\left(\frac{\theta}{2}\right) + 2r_2 \cos\left(\frac{\theta}{2}\right) + 2r_3 \cos\left(\frac{\theta}{2}\right) + \cdots + 2r_{n-1} \cos\left(\frac{\theta}{2}\right) + r_n \cos\left(\frac{\theta}{2}\right)$$

The graphic below illustrates ten circles inside a right triangle. These ten circles consume 77.072 % of the triangle's area.



Conclusion: The impact of the Pythagorean Theory on modern western thought can't be understated. While the Pythagorean relationship was not actually derived by Pythagoras, and was known as early as 1800 BC in ancient Babylon, it was through Euclidean Geometry and the techniques and theorems of early Greek mathematicians that the need for rigorous proof in all aspects of life was first understood. Note how often the Pythagorean Theorem was used in proving the basic QM > AM > GM > HM inequality. Many basic principles of math & life are based on the Pythagorean Theorem.